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An Adjustment to Brass's Logit Model of the Probabilities of Survival

Introduction

THE linear model involving the logits of the survivorship functions of any two life tables was proposed by Brass in 1975, and it has become very well known all over the world. The high correlation, usually in the neighbourhood of .99, made the bivariate linear regression of one logit on another very useful in estimating one life table functions from any other. Recently, it has been shown (Mitra, 1995) that the model, as it is, fails to meet an important boundary condition unless the slope coefficient of the linear equation happens to be equal to one. Forcing such a restriction on the model lowers its power of reproducibility by a large amount. Accordingly, a search for alternative models, or a modification of the present model seems to be warranted. The former has been attempted earlier (Mitra, 1983, 1995; Mitra and Levin, 1993; Mitra and Denny, 1994) and in this paper we have tried to investigate the latter. We begin first by looking into the theoretical or substantive argument Justifying the strong empirical relationship between the logits of the survivorship functions of any two-life tables.

The Logit Model

Brass *(ibid.)* noted, and rightly so that the logit transformation of the survivorship function $l(x)$, given by

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$$\text{logit } l(x) = 1/2 \ln \left\{\frac{1 - l(x)}{l(x)}\right) \qquad (1)$$

results in a variable that ranges from -∞ to +∞ as against the very small range of variation of 0 to 1 *of l(x).* Citing an example he proposed that this transformation also produces a function that varies linearly with age *x.* The logic has been shown to be erroneous (Mitra, 1995) and accordingly the justification of the strong linear relationship between the two logits, which is an empirical fact, must be sought elsewhere. In that endeavour, we begin by offering the following simple description of the pattern of variation of *l(x)* with *x.*

We suggest that the rate of change in /(x) at any age can be regarded as proportional to

(1) itself or /(x),
(2) its complement 1 - *l(x)*, which is the probability of dying in the age interval (0, *x)*, and finally,
(3) the reciprocal of age *x* or *lx.*

Needless to say, the utility of each of these components in describing the pattern of variation can be theoretically justified. For example, the rate of change in *l(x)* is expected to be high if there are few survivors, i.e., when *l(x) is* small or 1 *-l(x)* is large. In the same way, the rate of change will be small if *l(x)* is large or 1 - *l(x)* is small. These two components can be combined into one if *l* (x) is multiplied by 1 - *l(x).* However, that as such will make the rate of change equal to zero when *l(x)* =1 at *x* = 0 which is not true. In fact, it is known to be quite large at start of the *l(x)* curve. Accordingly, incorporation of an additional factor in the form of the reciprocal *of x* will make the derivative nonzero and allow it to assume large or small absolute values depending on the nature of its remaining unknown traits. Combining all of them, we can express the pattern of variation of *l* (x) in the form of a differential equation as

$$\frac{dl(x)}{dx} = \frac{l(x)(1-l(x))}{x} (b + Cg(X)) \qquad (2)$$

where *b* and c are suitable constants and *b + cg(x)* account for all other components of variation other than the three mentioned above. The constants *b* and c could have been absorbed by *g(x)* but have been separated for mathematical convenience that will soon become clear.

The solution of the differential equation (2) can be easily obtained by first writing it as

$$\frac{dl(x)}{l(x)(1-l(x))} = \left(\frac{b}{x} + \frac{cg(x)}{x}\right) dx \tag{3}$$

Integrating both sides of (3) we get

$$-ln\left(\frac{1-l(x)}{l(x)}\right) = a + blnx + ch(x) \tag{4}$$

For reasons of simplicity let us assume that $h(x)$ remains the same for all life tables but the three parameters a, b, and c may assume different values for different life tables. That is to say for another life table with survivorship function $l^*(x)$ we can write

$$-ln\left(\frac{1-l^*(x)}{l^*(x)}\right) = a^* + b^*lnx + c^*h(x) \tag{5}$$

Equations (4) and (5) can be combined by multiplying the latter by c/c^* and subtracting from the former to obtain

$$-ln\left(\frac{1-l(x)}{l(x)}\right) = a - a^*\frac{c}{c^*} + \left(b - b^*\frac{c}{c^*}\right) lnx - \frac{c}{c^*} ln\left(\frac{1-l^*(x)}{l^*(x)}\right) \tag{6}$$

Note that, the left hand sides of (4) and (5) are -2 logit $(l(x))$ and -2 logit $l^*(x)$, respectively. From (6), therefore, it follows that the modification in the Brass model suggested by this approach is that the logit of $l(x)$ of one table be regarded as a linear function not only of the logit of $l(x)$ of another but also of lnx.

At this point let us look into the boundary condition that was mentioned but not elaborated earlier. To that end, let us first write (6) as

$$ln\left(\frac{1-l(x)}{l(x)}\right) = B(0) + B(1)\ lnx + B(2)\ ln\left(\frac{(1-l^*(x)}{l^*(x)}\right) \tag{7}$$

where the B coefficients have been used to replace the corresponding coefficients of (6) for notational simplicity. Now differentiating both sides of (7) and simplifying, we get

$$\frac{\mu(x)}{1-l(x)} = \frac{B(1)}{x} + B(2)\frac{\mu^*(x)}{1-l^*(x)} \tag{8}$$

in which

$$\mu(x) = -\frac{1}{l(x)}\frac{dl(x)}{dx} = -\frac{dlnl(x)}{dx} \tag{9}$$

is the force of mortality at age x. Next, multiplying both sides (8) by $1 - l(x)$ we get

$$\mu(x) = B(1)\frac{1 - l(x)}{l(x)} + B_2\mu^*(x)\frac{1 - l(x)}{1 - l^*(x)} \tag{10}$$

The relationship described by (10) is supposed to hold for all values of x. However, for $x = 0$, it assumes an indeterminate form as $l(0) = 1$. The problem can of course be easily resolved by the application of L' Hospitals rule available for such cases. Accordingly, the limiting form of (10) as $x \rightarrow 0$ becomes

$$\mu(0) = B(1)\mu(0) + B(2)\mu(0) \tag{11}$$

from which the condition

$$B(1) + B(2) = 1 \tag{12}$$

is obtained. Note that since Brass's model did not include the variable lnx, the same exercise would produce the requirement that $B(2)$ become equal to one, as was mentioned earlier.

Next, let us rewrite (7) by taking (12) into account as

$$u(x) = B(0) + (1 - B(2))lnx + B(2)u^*(x) \tag{13}$$

in which we write $u(x)$ for $ln\left(\frac{1 - l(x)}{l(x)}\right)$ for notational convenience. Similar definition holds for $u^*(x)$ as well. Equation (13) can therefore be looked upon as a linear model with two parameters based on the logits of any two life tables. Observe that it can be written more explicitly and in a compact form as

$$v(x) = B(0) + B(2)v^*(x) \tag{14}$$

where

$$v(x) = u(x) - lnx \tag{15}$$

$$= ln\left(\frac{1 - l(x)}{l(x)}\right) - lnx \tag{16}$$

and similarly for $v^*(x)$.

Additional Observations

At this point we would like to return to equation (7) from which our restricted model equation (13) followed by (14) were derived because of (12). The reader may be wondering why we did not look into a model equation parallel to (14) by first writing (7) by virtue of (12) as

$$u(x) = B(0) + B(1)lnx + (1 - B(1))u^*(x) \tag{17}$$

instead of (13) and then simplify (17) as

$$w(x) = B(0) + B(1)y(x) \tag{18}$$

where

$$w(x) = u(x) - u^*(x) \tag{19}$$

and

$$y(x) = lnx - u^*(x) \tag{20}$$

The structural difference between (14) and (18) is worth noting. In (14), $v^*(x)$ can be written from $v(x)$ by changing $l(x)$ to $l^*(x)$ while a similar pattern can not be found when we look into $w(x)$ and $y(x)$. Accordingly, we have chosen to write $y(x)$ instead of $w^*(x)$ in (20), and have labelled (18) as model MO(2A) for future reference. Note that (19) and (20) can also be written as

$$w(x) = v(x) - v^*(x) \tag{21}$$

and

$$y(x) = -v^*(x) \tag{22}$$

The question that naturally arises is whether there is any difference between MO(2) and MO(2A) with respect to their abilities to reproduce the variable of interest, namely, a life table survivorship function $l(x)$. Unfortunately, since the dependent variables of these two models are different, a straight forward comparison of the two can not be made. However, noting that the residual sum of squares of both the models are equal we may try an indirect way of doing so by comparing their respective coefficients of determination. In this example, they are the squares of the bivariate correlations between $v(x)$ and $v^*(x)$ and between $w(x)$ or $v(x) - v^*(x)$ and $v^*(x)$, which we shall denote by $r^2(v, v^*)$ and $r^{*2}(v - v^*, v^*)$ respectively.

According to the standard formula connecting the slope and the correlation coefficient of a bivariate regression, $B(2)$ of (14) and $B(1)$ of (18) can be expressed as

$$B(2) = r(v, v^*) \frac{\sigma(v)}{\sigma(V^*)} \tag{23}$$

$$-B(1) = r(v - v^*, v^*) \frac{\sigma(v - v^*)}{\sigma(v^*)} \tag{24}$$

where σ stands for the standard deviation.

Squaring both sides of (24), expanding $\sigma^2(v - v^*)$ and simplifying, we get

$$B^2(1) = r^2(v - v^*, v^*) \left(1 + \frac{\sigma^2(v)}{\sigma^2(v^*)} - 2r\,(v, v^*) \frac{\sigma(v)}{\sigma(v^*)}\right) \tag{25}$$

which due to (23) can be written as

$$B^2(1) = r^2(v - v^*, v^*) \left(1 + \frac{B^2(2)}{r^2(v, v^*)} - 2B(2)\right) \tag{26}$$

$$= r^2(v - v^*, v^*) \left((1 - B(2))^2 + B^2(2) \left(\frac{1}{r^2(v, v^*)} - 1\right)\right)$$

which

$$= r^2(v - v^*, v^*) \left(B^2(1) + B^2(2) \left(\frac{1}{r^2(v, v^*)} - 1\right)\right) \tag{27}$$

due to (12). Rearranging terms of (27), we get

$$\frac{1}{r^2(v - v^*, v^*)} - 1 = \frac{B^2(2)}{B^2(1)} \left(\frac{1}{r^2(v, v^*)} - 1\right) \tag{28}$$

Therefore,

$$r^2(v, v^*) \gtrless r^2 (v - v^*, v^*) \tag{29}$$

according as

$$B^2(2) \gtrless B^2(1) \tag{30}$$

Note that the inequality (30) can also be written as

$$B(2) \gtrless ½ \tag{31}$$

due to (12). The choice between MO(2) or MO(2A) can therefore be easily made by running both regression and then comparing their slope coefficients.

Estimates of Parameters and the Quality of the Model

As is customary with estimating model parameters, we have, in the following, tried the method of least squares of errors. It should be noted that the associated R^2 generated by the method is not to be interpreted in the statistical sense as the square of a correlation coefficient found between two random variables. Rather it should be treated more like the coefficient of determination or as a measure of the goodness of fit. What follows next is the description of this exercise, for which we have arbitrarily selected six life tables from Keyfitz and Flieger (1971), formed three pairs and within each pair treated the life table $l(x)$ function of one as the dependent and the other as the independent variable of the standard regression equation. The R^2 measure of goodness of fit has been obtained for every pair for each of the three models namely,

(1) Brass's original logit model with two parameters,
(2) the modified model with an additional variable and three parameters given by (7), and
(3) the modified model with an additional variable and two parameters given by (14).

Values of the measure may be seen in Table 1 where for operational simplicity we have labelled the three models as BR(2), MO(3), and MO(2) respectively (BR for Brass, MO for modified, the numbers 2 and 3 for the number of parameters). It may be mentioned at this point that all four pairs of countries used to generate Table 1 turned out to have *B(2)* values in the neighbourhood of 1 and accordingly we chose not to proceed with the additional computations required for MO(2A).

TABLE 1: R^1 VALUES OF THREE PAIRS OF COUNTRIES BY THREE MODELS

Based on the Male l(x)	*Values of Countries*	R^1 *Values Corresponding to Model*		
Dependent	*Independent*	*BR(2)*	*M0(3)*	*M0(2)*
Portugal (1966-68)	Sweden (1967)	.9702	.9730	.9898
Dominican Republic (1966)	Canada (1966-68)	.9906	.9906	.9829
Japan (1966)	Ceylon (1967)	.9950	.9952	.9962
Indonesia	Philippines	.9908	.9989	.9979

Since the model M0(3) was obtained by adding a second independent variable to the model BR(2), it follows that the former will produce a fit at least as good as the latter. The R^2 measure of goodness of fit in Table 1 indeed shows that the difference between the two is in the right direction but it is very small. This is so mainly because in all three examples the R^2 values associated with BR(2) are already very close to the upper limit of 1 which leaves very little scope for further improvement.

In the same way, it may be argued that the two parameter model as presented in (13) will have R^2 values no higher than the same obtained for M0(3). However, it should be noted that its interpretation for (13) is different from that which can be made for M0(2) given by (14). This is due to the fact, that the model expressed as (14) has a different dependent variable (see eqns. 15 and 16), namely, $v(x)$ as against $u(x)$ of (13). It is for this reason that two of the four pairs of life tables have produced for M0(2) R^2 values which are even larger than those of the other two models. In one example, R^2 for M02 is greater than that for BR(2) but less than that for M0(3). The other pair for which it is less than BR(2), difference is quite small. Be that as it may, it is obvious that as a two parameter model, M0(2). fits the data quite well but most of all, it meets the important boundary condition. It is not known what the R^2 value would have been if values of $l(x)$ for $0 < x < 1$ could be included in the model. However, it can be said that failure to meet the boundary condition would almost certainly affect the performance of BR(2) by a nonnegligible amount.

Next we present the results of another experiment in which one standard country namely, Philippines with male life expectancy of 50 years, has been used to generate the estimates of several others. As before, for all these countries $B(2)$ values were greater than 1/2 and therefore, like Table 1, we present the results for the same three models in Table 2.

TABLE 2: R^2 VALUES OF SEVERAL COUNTRIES BY THREE MODELS USING PHILIPPINE (1960) AS THE STANDARD (MALES ONLY)

Country	*R^2 Corresponding to Model*		
	BR(2)	*MO(3)*	*MO(2)*
Ceylon (1967)	.9909	.9954	.9930
Hong Kong (1966)	.9904	.9992	.9956
Israel (1967)	.9886	.9972	.9940
Sarawak (1961)	.9956	.9982	.9978
Cameroon (1964)	.9847	.9947	.9963
Mad (1966)	.9539	.9959	.9830
Guatemala (1964)	.9964	.9969	.9758

As may be seen from Table 2, for all of those countries, the R^2 values for MO(2) are all greater than those for BR(2). Similar results were obtained by using other countries with life expectancies higher as well as lower than that of Philippines.

Summary and Concluding Remarks

Brass proposed the logit transformation of the life table survivorship function in formulating his linear model. The model was found quite satisfactory for estimation purposes and that provided its justification more than the reason Brass cited in favour of the transformation. Some of the deficiencies of the model resulting from its failure to meet a boundary condition has been noted in later studies which led to the development of other models with encouraging results. In this paper, a theoretical justification has been provided to support the logit transformation which together with the boundary condition mentioned before have given rise to important and interesting modification of Brass's model. In its present form, the logit of $l(x)$ of one life table has been expressed not only as a linear function of the same of another life table (which is Brass'smodel), but also of another variable, namely the logarithm of age. The restriction imposed by the boundary condition however, has reduced the number of parameters of the model from three to two. That by itself is a desirable characteristic of the model but in addition, the quality of the fit shown by the preliminary investigation reported in this study has been found satisfactory enough to justify further investigation along this line.

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