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On a Renewal Theory Approach for Estimating the Parity Specific Fertility Rates under Intrinsic Variation of Fecundity Level

Introduction

THE limitation of fertility analysis lies in not considering all the pertinent factors together governing the fertility behaviour. Biswas (1978, 1980) have considered the waiting time distribution for the nth birth as the convolution of several Poisson distribution with variable parameters λ_i $(i = 1, 2, \ldots, n)$ where λ_i stands for the hazard rate of the ith parity. This has given rise to density function of the waiting time as

$$f_n(s_n \mid \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \lambda_i e^{-\lambda_i s_n} \tag{1}$$

where

$$\lambda_1 = \lambda, \lambda_2 = \lambda e^{-\delta}, \lambda_3 = e^{-2\delta}, \ldots, \lambda_n = \lambda e^{-(n-1)\delta} \tag{2}$$

and $\quad s_n = \sum_{i=1}^{n} t_i.$

While the parity specific fertility (hazard) rates are considered in the foregoing model the individual differences are not taken into account. However, Brass (1958), Singh (1964) have assumed the variation of fecundability parameter λ given by

$$\phi(\lambda) = \frac{a^k}{\sqrt{(k)}} e^{-a\lambda} \lambda^{k-1} \; ; \; 0 \leq \lambda < \infty \quad a, k < \infty \tag{3}$$

This gives the waiting time distribution of nth birth as weighted Poisson process with gradually diminishing intensity given by

$$f_n(s_n) = \frac{a^k}{\sqrt{(k)}} \int_0^{\infty} \sum_{i=1}^{n} \prod_{j \neq i} \frac{e^{-(j-1)\delta - (i-1)^{\delta}}}{[e^{-(j-1)\delta} - e^{-(i-1)\delta}]} \times e^{-\lambda [e^{-(i-1)\delta} s_n + a]} \lambda^k \, d\lambda \tag{4}$$

$$= \frac{ka^k}{\sqrt{(k)}} \sum_{i=1}^{n} \prod_{j \neq i} \frac{e^{-\delta(i+j-2)}}{[e^{-(j-1)\delta} - e^{-(i-1)\delta}]} \times \frac{1}{(a + e^{-(i-1)\delta} s_n)^{k+1}}. \tag{5}$$

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The raw moments of the above distribution are given by

$$\mu_r' = E(s_n^r) = \sum_{i=1}^{n} \prod_{j\neq i} \frac{e^{-\delta(i+j-2)}}{[e^{-(j-1)\delta} - e^{-(i-1)\delta}]} \times ka^k \int_0^\infty \frac{s_n^r \, ds_n}{(a + e^{-(i-1)\delta} \, s_n)^{k+1}}$$

$$= \sum_{1=1}^{n} \prod_{j\neq i} \frac{e^{-\delta(i+j-2)}}{[e^{-(j-1)\delta} - e^{-(i-1)\delta}]} \, e^{\delta(i-1)(r+1)} \, a^{r-k} \beta(r+1, k-r). \quad (6)$$

An attempt has also been made in this paper to generalise the assumption of uniform declination of fertility rate by parity to a non uniform decrease of the same as observed from the data. Thus hazard rates are therefore taken as

$$\lambda_1 = \lambda$$

$$\lambda_2 = \lambda_1 e^{-\delta_1} = \lambda e^{-\delta_1}$$

$$\lambda_3 = \lambda_2 \, e^{-\delta_2} = \lambda e^{-\delta_1} \, e^{-\delta_2} = \lambda e^{-(\delta_1+\delta_2)}$$

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$$\lambda_i = \lambda_{i-1} \, e^{-\delta_{i-1}} = \lambda e^{-\delta_1} \, e^{-\delta_2} \ldots e^{-\delta_{i-1}} = \lambda e^{-(\delta_1+\delta_2+\ldots+\delta_{i-1})} \quad (7)$$

Note that for $\delta_1 = \delta_2 = \ldots = \delta_i$. The above hazard rates reduce to the hazard rates of Biswas (1978).

Further, λ varies from individual to individual conforming a two parameter family of Gamma Distribution. This gives the waiting time density function for the nth birth as

$$f_n(s_n) = \frac{a^k}{\sqrt{(K)}} \int_0^\infty \sum_{i=1}^{n} \prod_{j\neq i} \frac{e^{-(\delta_1+\ldots+\delta_{j-1})-(\delta_1+\ldots+\delta_{i-1})}}{e^{-(\delta_1+\ldots+\delta_{j-1})} - e^{-(\delta_1+\ldots+\delta_{i-1})}} \times$$

$$e^{-\lambda[e^{-(\delta_1+\ldots+\delta_{i-1})} s_n + a]} \, \lambda^k \, d\lambda \quad (8)$$

$$= \frac{ka^k}{\sqrt{(k)}} \sum_{i=1}^{n} \prod_{j\neq i} \frac{e^{-(\delta_1+\ldots+\delta_{j-1})} \, e^{-(\delta_1+\ldots+\delta_{i-1})}}{e^{-(\delta_1+\ldots+\delta_{j-1})} - e^{-(\delta_1+\ldots+\delta_{i-1})}} \times$$

$$\frac{1}{(a + e^{-(\delta_1+\ldots+\delta_{i-1})} \, s_n)^{k+1}} \quad (9)$$

and moments of the distribution will be

$$\mu_r' = E(s_n^r) = \sum_{i=1}^{n} \prod_{j\neq i} \frac{e^{-(\delta_1+\ldots+\delta_{j-1})} \, e^{-(\delta_1+\ldots+\delta_{i-1})}}{e^{-(\delta_1+\ldots+\delta_{j-1})} \, e^{-(\delta_1+\ldots+\delta_{i-1})}} \times$$

$$e^{(\delta_1+\delta_2+\ldots+\delta_{i-1})(r+1)} \, a^{r-k} \beta(r+1, k-r). \quad (10)$$

Data and Analysis

The data utilized for testing the model relates to the Fertility Trend in Delhi.

A survey was conducted by Population Research Centre, Institute of Economic Growth, for Delhi Administration, Directorate of Health and Family Welfare in the year 1987.

The Whole Delhi was divided into six zones. Each zone is further divided into several blocks. The blocks are chosen in the same manner as that considered in Census 1981, e.g. if one zone constitutes 30% of the population size, then sample size is determined accordingly. On a whole, a sample of 2114 households has been collected. The first house of the block is chosen by random number table and thereafter every 10th house is picked up for collecting the information for all married couples in the house belonging to age 45 and below.

A further sub-sample has been chosen by taking every fifth household of 2114 household already collected, so that a total of 415 households are covered.

Let t_i is the waiting time between births and T is the total reproductive time, then

$$\overline{0 \qquad t_1 \qquad t_2 \ldots t_n \qquad T}$$

$$L = \lambda_1^{n_1} e^{-\lambda_1 \sum_{i=1}^{n} t_{1i}} \lambda_2^{n_2} e^{-\lambda_2 \sum_{i=1}^{n} t_{2i}} \ldots \lambda_k^{n_k} e^{-\lambda_k \sum_{i=1}^{n} t_{ki}} \tag{11}$$

From the data of 415 cases. We found 375 has gone to 1st parity. Therefore,

$$n_1 = 375$$

and sum of months

$$\sum_{i=1}^{n_1} t_{1i} = 9649.$$

From (11), we have

$$\lambda_1 = \frac{375}{9649} = .0382 \text{ months}$$

$$= .4664 \text{ years.}$$

Similarly

$$\lambda_2 = .4098 \text{ yrs.}$$

$$\lambda_3 = .3351 \text{ yrs.}$$

$$\lambda_4 = .3928 \text{ yrs. and so on}$$

giving $\delta e^{-\delta_1} = .8786$ and $e^{-\delta_2} = .8177$.

Now for $n = 3$

$$\mu_r' = E(s_3^r)$$

$$= \left[\frac{e^{-\delta_1}}{e^{-\delta_1}} \frac{e^{-(\delta_1+\delta_2)}}{e^{-(\delta_1+\delta_2)}}\right] a^{r-k\beta(r+1,k-r)}$$

$$+ \left[\frac{e^{-\delta_1}}{-e^{-\delta_1}} \frac{e^{-(\delta_1+\delta_2)}e^{-\delta_1}}{e^{-(\delta_1+\delta_2)}e^{-\delta_1}}\right] e^{\delta_1(r+1)} \; a^{r-k\beta(r+1,k-r)}$$

$$+ \left[\frac{e^{-(\delta_1+\delta_2)}}{-e^{-(\delta_1+\delta_2)}} \frac{e^{-\delta_1}e^{-(\delta_1+\delta_2)}}{e^{-\delta_1}-e^{-(\delta_1+\delta_2)}}\right] e^{(\delta_1+\delta_2)(r+1)} \; a^{r-k\beta(r+1,k-r)}.$$

Substituting $e^{-\delta_i^{\wedge}}$ and $\lambda_i^{\wedge s}$, gives

$$\mu_1' = 1.2968\, a^{1-k\beta(2,k-1)}$$

and $$\mu_2' = 2.645\, a^{2-k\beta(3,k-2)} \quad (12)$$

The computed μ_1' and μ_2' from the data are

$$\mu_1' = 25.73709 \text{ months} = 2.1442 \text{ years}$$

and $$\mu_2' = 29.2848 \text{ months} = 2.4404 \text{ years.}$$

Thus

$$2.1442 = 1.2968\; a^{1-k\beta(2,k-1)}$$

$$2.4404 = 2.645\; a^{2-k\beta(3,k-2)}$$

Taking $\theta_1 = 1 - k\beta(2, k-1)$

and $\theta_2 = 2 - k\beta(3, k-2)$

Gives

$$a^{\theta_1} = 1.6535$$

$$a^{\theta_2} = 0.9226 \quad (13)$$

Solving for θ_1 and θ_2, the estimates of k and a are obtained as

$$k = 2.6037$$

and $$a = 3.804.$$

Application of the Model

The parameters estimated above i.e. $\hat{a}$ and $\hat{k}$ are substituted in the model (9) to obtain

$$f_n(s_n) = \frac{(2.6037)\,(3.804)^{2.6037}}{\sqrt{(2.6037)}} \sum_{i=1}^{n} \prod_{j \neq i} \frac{e^{-(\delta_1 + \ldots + \delta_{j-1})}\; e^{-(\delta_1 + \ldots + \delta_{i-1})}}{e^{-(\delta_1 + \ldots + \delta_i - 1)} - e^{-(\delta_1 + \ldots + \delta_{j-1})}}$$

$$\frac{1}{(3.804 + e^{-(\delta_1 + \ldots + \delta_{i-1})}\, s_n)^{3.6037}}. \tag{14}$$

It can be immediately seen that the waiting time distribution of s_n for $n = 1$ i.e. s_1 is given by

$$f_1(s_1) = \frac{ka^k}{\sqrt{(k)}} \frac{1}{(a - 1s_n)^{k+1}} \tag{15}$$

Thereafter taking into consideration that the hazard rate during first to second parity is changed from λ to $\lambda e^{-\delta_1}$ where λ conforms to

$$\phi(\lambda) = \frac{a^k e^{-a\lambda} \lambda^{k-1}}{\sqrt{(k)}} \qquad 0 < \lambda < \infty$$

$$a, k > 0. \tag{16}$$

The distribution of s_n for $n = 2$ (i.e. s_2) is given by

$$f_2(s_2) = \frac{a^k}{\sqrt{(k)}} \int_0^\infty \lambda e^{-\delta_1} e^{-\lambda e^{-\delta_1} t} e^{-a\lambda} \lambda^{k-1}\, d\lambda$$

$$= \frac{ka^k e^{-\delta_1}}{(a + e^{-\delta_1} t)^{k+1}}. \tag{17}$$

Precisely in a similar way the distribution of s_n for $n = 3$ is given by

$$f_3(s_3) = \frac{a^k}{\sqrt{(k)}} \int_0^\infty \lambda e^{-(\delta_1 + \delta_2)} e^{-\lambda e^{-(\delta_1 + \delta_2)} t} e^{-a\lambda} \lambda^{k-1} d\lambda = \frac{ka^k e^{-(\delta_1 + \delta_2)}}{(a + e^{-(\delta_1 + \delta_2)} t)^{k+1}} \tag{18}$$

Finally, the distribution of s_n is given by

$$f_n(s_n) = \frac{a^k}{\sqrt{(k)}} \int_0^\infty \lambda e^{-(\delta_1 + \delta_2 + \ldots + \delta_n)} e^{-\lambda e^{-(\delta_1 + \ldots + \delta_n)} t} e^{-a\lambda} \lambda^{k-1} d\lambda$$

$$= \frac{ka^k e^{-(\delta_1 + \delta_2 + \ldots + \delta_n)}}{(a + e^{-(\delta_1 + \delta_2 \ldots + \delta_n)} t)^{k+1}}. \tag{19}$$

Accordingly for s_2 the proportion of women having 2nd birth between x to $(x + 1)$ years is given by

$$\int_x^{x+1} f_2(s_2)\, ds_2 = a^k \left[\frac{1}{[a + e^{-\delta_1} x]^k} - \frac{1}{[a + e^{-\delta_1} (x + 1)]^k} \right], \tag{20}$$

and that between x to $(x + 1)$ years for s_3 is given by

$$\int_{x}^{x+1} f_3(s_3)\,ds_3 = a^k \left[\frac{1}{[a + e^{(-\delta_1 + \delta_2)}\, x]^k} - \frac{1}{[a + e^{(-\delta_1 + \delta_2)}\,(x+1)^k\,]} \right] \qquad (21)$$

and so on.

The fertility table for 1st, 2nd and 3rd parity is shown as under

FERTILITY TABLE

Time\Parity	*1st*	*2nd*	*3rd*
0-1	.4554	.4178	.3598
1-2	.2107	.2256	.2042
2-3	.1135	.1183	.1232
3 and above	.2204	.2383	.3128

Discussion

The model is pertinent for estimating parity specific fertility for a compound population exhibiting different levels of intrinsic fecundity level. Therefore, one possible application of this study lies in constructing fertility table based on the renewal theoretic model which we have developed. The premises of the model lies in systematic reduction of parity specific fertility (hazard) rates subject to inherent variation in the fecundity level. A more general model could have been by considering the age of the women along with the parity taking into consideration of the variability in the fecundity level. However, since parity and age are highly correlated at least in lower and middle level socio-economic set up (especially with the data which we have employed for the analysis) hardly much benefit is expected by introducing age also as a variable at the expenses of more Mathematical Complications. Still with a large scale data covering all the socio-economic strata of a population one could test the present model more effectively.

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