Demography India
Vol. 18 (1989), pp. 81-93

S. Mitra

Estimates of the Parameters of the Trajectory of Births

1. The Problem

The equation of the birth trajectory generated by Lotka's stable population model is traditionally expressed as

$$B(t) = \sum_{j=0}^{\infty} Q_j e^{r_j t} \tag{1}$$

where $B(t)$ is the number of births at time t and r_j ($j = 1, 2, \ldots$) are the roots of r from the integral equation

$$\int_0^{\infty} e^{-rx} \phi(x)\, dx = 1 \tag{2}$$

in which $\phi(x)$ stands for the net maternity rate at age x. The roots r_j of (2) are all complex except for one, say r_o, which is real. The parameters Q_j are given by (see Keyfitz 1968)

$$Q_j = \frac{\int_0^{\infty} e^{-r_j t} G(t)\, dt}{\int_0^{\infty} x e^{-r_j x} \phi(x) dx} = \frac{G^*(r_j)}{m_j} \tag{3}$$

where $G(t)$ is the number of births at time t to those who were alive at the start of the process, i.e. at $t = 0$. The integral in the numerator of (3) is also known as the one sided Laplace transform of $G(t)$ i.e. $G^*(r_j)$ with the parameter r_j. The denominator of (3) is the mean age of childbearing m_j obtained from the frequency function given by (2) for $r = r_j$. Denoting the upper boundary of the reproductive interval by β, we can write

$$G(t) = 0 \text{ for } t \geq \beta \tag{4}$$

since no more children are born to the initial population after β years.

For the computation of *Qj,* the method of numerical integration is usually resorted to after the values of *G(t)* are obtained by projecting the births of the initial population over the time interval (0, β). The procedure is quite cumbersome and especially so when *rj is* complex, The purpose of this paper is to develop an analytic procedure to achieve the same end by assuming a simple but a plausible shape of the *G(t)* function that would make its Laplace transform amenable to straightforward mathematical manipulations. Subsequently, we shall substitute an analytic expression of the mean age of childbearing *m;* developed earlier (Mitra and Levin 1990) in the denominator of (3) in an attempt to develop an operationally convenient formula for *Qi,*.

2. Formulation of the *G(t)* Function

We begin by noting that *G(t)* is likely to increase for some time after the start of the process at $t = 0$, especially when the population is increasing. This is so because the size of the reproductive population out of those alive at $t = 0$ will continue to grow until $t = \alpha$, the minimum age of the reproductive interval. Thereafter, the size of this population will continue to decline together with *G(t).*

For reasons of simplicity, a model for patterning the- variation of G(t) may be described in terms of two line segments meeting at $t - \alpha$. The first segment starts with B(t) = B(0) at $t = 0$ with a slope, of k, the value of which has to be determined in an appropriate manner. Its equation can, therefore, be written as

$$G(t) = B(0)\,(1 + kt),\ \ 0 \leq t \leq \alpha \tag{5}$$

The second line segment begins with $G(\alpha)$ at $t = \alpha$ that declines linearly for increasing values of t in a way such that $G(\beta) = 0$ for $t = \beta$. The equation of this line may be written as

$$G(t) = \frac{B(0)\,(1 + k\alpha)\,(\beta - t)}{\beta - \alpha},\ \ \alpha \geq t \geq \beta \tag{6}$$

As it stands, the function $G(t)$ defined by (5) and (6) is continuous in the interval $0 \geq t \geq B$. It is also differentiable at every point in that interval except at $t = \alpha$. That is sufficient for the derivation of the one-sided Laplace transform $G^*(r)$ of $G(t)$, i.e. the numerator of (3) to which we turn next.

3. Derivation of $G^*(r)$

Substitution of (5) and (6) in the numerator of (3) gives, after dropping the subscript j,

$$G^*(r) = B(0)\ \left[\int_0^{\alpha} e^{-rt}(1+kt)\,dt + \frac{1+k\alpha}{\beta-\alpha}\int_{\alpha}^{\beta} e^{-rt}(\beta - t)\,dt\right] \tag{7}$$

The first integral in (7), i.e.

$$\int_0^{\alpha} e^{-rt}(1+kt)\,dt = -(1+kt)\frac{e^{-rt}}{r}\Bigg]_0^{\alpha} + k\int_0^{\alpha}\frac{e^{-rt}}{r}\,dt$$

by partial integration which then becomes

$$= \frac{1-e^{-r\alpha}(1+k\alpha)}{r}\ \frac{k}{r}\left(\frac{1-e^{-r\alpha}}{r}\right) \tag{8}$$

In the same way, the second integral in (7), i.e.

$$\int_{\alpha}^{\beta} e^{-rt}(\beta - t)\,dt = \frac{(\beta-\alpha)\,e^{-r\alpha}}{r} - \frac{e^{-r\alpha}-e^{-r\beta}}{r^2} \tag{9}$$

Substitution of (8) and (9) in (7) and subsequent simplification produces

$$G^*(r) = B(0)\left[\frac{1}{r} + \frac{k(1-e^{-r\alpha})}{r^2} - \frac{(1+k\alpha)\,(e^{-r\alpha}-e^{-r\beta})}{(\beta-\alpha)r^2}\right] \tag{10}$$

which can be alternatively expressed as

$$G^*(r) = B(0)\left[\frac{1}{r} + \frac{k}{r^2} - \frac{(1+k\beta)\,e^{-r\alpha} - (1+k\alpha)\,e^{-r\beta}}{(\beta-\alpha)r^2}\right] \tag{11}$$

When r is complex, i.e.

$$r = u + iv \tag{12}$$

where u and v are real, (11) can be expressed as a complex number like (12) by substituting $u + iv$ for r in (11). Thus,

$$\frac{1}{r} = \frac{1}{u+iv} = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2} \tag{13}$$

is similar in form as that of (12). Similarly,

$$\frac{1}{r^2} = \frac{1}{u^2 + 2iuv - v^2} = \frac{u^2 - v^2}{(u^2 - v^2)^2 + 4u^2v^2} - i\frac{2uv}{(u^2 - v^2)^2 + 4u^2v^2}$$

$$= \frac{u^2 - v^2}{(u^2 + v^2)^2} - i\frac{2uv}{(u^2 + v^2)^2} \tag{14}$$

Next,

$$\frac{e^{-r\alpha}}{r^2} = \frac{e^{-u\alpha}(\cos(v\alpha) - i\sin(v\alpha)(u^2 - v^2 - 2iuv)}{(u^2 + v^2)^2}$$

$$= \frac{e^{-u\alpha}}{(u^2 + v^2)^2}\left[(u^2 - v^2)\cos(v\alpha) - 2uv\sin(v\alpha)\right] -$$

$$\frac{ie^{-u\alpha}}{(u^2 + v^2)^2}\left[(u^2 - v^2)\sin(v\alpha) + 2uv\cos(v\alpha)\right] \tag{15}$$

and the last one i.e. $e^{-r\beta} / r^2$ is the same as (15) in which α has to be replaced by β. Substitution of (13) through (15) in (11) gives G^* (r) in units of $B(0)$ as

$$G^*(r) = L + im \tag{16}$$

where $$L = \frac{u}{u^2 + v^2} + \frac{k(u^2 - v^2)}{(u^2 + v^2)^2}$$

$$- \frac{e^{-u\alpha}(1 + k\beta)}{\beta - \alpha} \cdot \frac{(u^2 - v^2)\cos(v\alpha) - 2uv\sin(v\alpha)}{(u^2 + v^2)^2}$$

$$+ \frac{e^{-u\beta}(1 + k\alpha)}{\beta - \alpha} \cdot \frac{(u^2 - v^2)\cos(v\beta) - 2uv\sin(v\beta)}{(u^2 + v^2)^2} \tag{17}$$

and $$M = \frac{v}{u^2 + v^2} + \frac{2kuv}{(u^2 + v^2)^2}$$

$$+ \frac{e^{-u\alpha}(1 + k\beta)}{\beta - \alpha} \cdot \frac{(u^2 - v^2)\sin(v\alpha) + 2uv\cos(v\alpha)}{(u^2 + v^2)^2}$$

$$- \frac{e^{-u\beta}(1+k\alpha)}{\beta-\alpha} \cdot \frac{(u^2-v^2)\sin(v\beta)-2uv\cos(v\beta)}{(u^2+v^2)^2} \tag{18}$$

For a given set of values of the parameters α, β, *u, v* and *k,* the values of L and *M* can be computed from (17) and (18) which in turn can be substituted in (16) for the determination of G*(r). At this point, it may be mentioned that since the complex roots appear with their conjugates there is an *u - iv* corresponding to every *u + iv.* It may be easily verified that corresponding to every such conjugate root there is a G*(r) which can be obtained from (16) by changing the sign of M.

4. Derivation of *Q*

All that remains to be determined for computing the value of *Q* is that of *m,* the denominator of (3), In an earlier study (Mitra and Levin 1990), a type III Pearsonian function used to describe the distribution of the net maternity rates θ*(x)* resulted in an analytic expression of mas

$$m = \alpha + \frac{\delta}{a+r} \tag{19}$$

where α is the start of the distribution of the net maternity function.

In (19) $$a = \frac{T-\alpha}{\sigma^2} \tag{20}$$

and $$\delta = \left(\frac{T-\alpha}{\sigma}\right)^2 \tag{21}$$

where T and σ are the mean and the standard deviation of the distribution of $\phi(x)$. When r is complex as in (12),

$$m = \alpha + \frac{\delta}{u+a+iv} = F + iG \tag{22}$$

Where $$F = \alpha + \frac{\delta(u+a)}{(u+a)^2+v^2} \tag{23}$$

and $$G = -\frac{\delta v}{(u+a)^2+v^2} \tag{24}$$

Substitution of (16) and (22) in (3) gives

$$Q = \frac{L + iM}{F + iG} = C + iD \tag{25}$$

where $$C = \frac{LF + MG}{F^2 + G^2} \tag{26}$$

and $$D = \frac{MF - LG}{F^2 + G^2} \tag{27}$$

It can be easily shown that the value of Q corresponding to the conjugate root $u - iv$ is $C\text{-}iD$.

These values of Q for different values of r may next be substituted in (1) for a complete determination of the birth trajectory. As is well known (Keyfitz 1968), the trajectory is real and should not have complex terms, even though Q is complex. This is so because the sum of the two terms involving any complex root and its conjugate, i.e.

$$(C + iD)\, e^{(u + iv)\, t} + (C - iD)\, e^{(u - iv)\, t}$$

$$= 2(C \cos (vt) - D \sin (vt))\, e^{ut} \tag{28}$$

is real as it should be. Thus denoting the complex roots by

$$r_j = u_j + iv_j, \text{ for } j = 1, 2, \ldots \tag{29}$$

We can express the birth trajectory given by (1) as

$$B(t) = Q_0 e^{r_0 t} + 2 \sum_{j=1}^{\infty} (C_j \cos (v_j t) - D_j \sin (v_j t))\, e^{u_j t} \tag{30}$$

in units of B(0) where Q_0 is the ratio of (17) and (22) for $u = r_0$ and $v = 0$.

Observe that each of the terms inside the summation sign represents a periodic function with ups and downs at regular intervals. These local optimums are reached when the derivative of a typical term with respect to t

i.e. $$\frac{d}{dt} (C \cos (vt) - D \sin (vt)\, e^{ut}) = 0 \tag{31}$$

For operational convenience we have dropped the subscript/ *in* (31). Differentiation of the function in (31) produces

$$u e^{ut} (C \cos(vt) - D \sin(vt)) - e^{ut}(C \sin(vt) + D \cos(vt)) = 0 \tag{32}$$

Dividing (32) by $e^{ut} \cos(vt)$ and simplifying we get

$$\tan (vt) = \frac{Cu - D}{C + Du} \tag{33}$$

Note that in (32) if $\cos(vt) = 0$ then $C + Du = 0$ as both $\sin(vt)$ and $\cos(vt)$ cannot be zero at the same time. That is consistent with (33) and therefore, the solution of t can be obtained from (33) as

$$\tau = \left[\tan^{-1} \left(\frac{cu - D}{C + Du} \right) + n\pi \right] / v$$

for those integral values of n for which $t \geq 0$. Noting from (33) that

$$\cos (v\tau) = \pm \frac{C + Du}{\sqrt{(C^2 + D^2)(1 + u^2)}} \tag{35}$$

and

$$\sin (v\tau) = \pm \frac{Cu - D}{\sqrt{(C^2 + D^2)(1 + u^2)}} \tag{36}$$

we can obtain the optimum values as

$$0 (\tau) = \pm 2 \sqrt{\frac{C^2 + D^2}{1 + u^2}} \quad e^{u\tau}$$

by substituting (35) and (36) in (30) where t is given by (34). The interval between any two successive local maximums (minimums) can also be determined from (34) as $2\pi/v$.

5. An Example

In the study cited earlier (Mitra 1990), a hypothetical example based on an average age of motherhood $T = 28$ years, a variance of the age distribution of motherhood $\sigma^2 = 39$ and a net reproduction rate R - *1*.25 produced $\alpha = 1/3$ and δ=13/3 when $\alpha = 15$ and $\beta = 45$ are taken as the boundaries of the reproductive interval.

For purposes of demonstration and for reasons of operational simplicity, the value of K has been set around .01 which is equivalent to a rate of growth of one percentage point. The reason for such a choice is that under normal conditions k should fluctuate around r which in this example is .008 or close to one percent. As a matter of fact it will be of some interest to see how sensitive G*(r) is to changes in r and *k*. The same question can also be asked with respect to alternative values of the reproductive interval (α,β). Accordingly, Table 1 has been designed to show the variations in G*(r) for different combinations of r, *k*, α and β where *r* is real.

TABLE 1 : VALUES OF G*fr) FOR SEVERAL COMBINATIONS OFα β, P, *k* AND *r*

k	r	Fertile Interval (α, β)								
		(10,45)	(10,50)	(10,55)	(12.5,45)	(12.5,50)	(12.5,50)	(15,45)	(15, 58)	(15,55)
008	.008	25.916	27.962	29.955	27.339	29.408	31.424	28.743	30.835	32.873
	.010	25.16	27.070	28.920	26.522	28.450	30.317	27.860	29.807	31.691
	.012	24.436	26.220	27.937	25.739	27.537	29.267	27.016	28.828	30.570
010	.008	26.306	28.390	30.420	27.823	29.939	32.001	29.319	31.468	33.561
	.010	25.536	27.482	29.366	26.988	28.960	30.870	28.415	30.414	32.349
	.012	24.799	26.617	28.366	26.189	28.028	29.797	27.551	29.410	31,200
012	.008	26.696	28.817	30.885	28.307	30.470	32.578	29.896	32.100	34.248
	.010	25.913	27.894	29.812	27.455	29.471	31.423	28.970	31.021	33.006
	.012	25.163	27.013	28.794	26.639	28.519	30.327	28.085	29.993	31.829

The values of G*(r) based on smaller increments of any one of these parameters with the other three held constant show a pattern of change that is virtually linear. Therefore, its value for any combination of values of the parameters *r, k,* α and β can be obtained to a reasonable degree of accuracy by straightforward linear interpolation or extrapolation as the case may be.

Accordingly, computations of the parameters *L, M,* etc. have been carried out only for the reproductive interval (15,45) that is used more often than any other. These have been done for the first few roots in decreasing order of *u* which may be seen in Table 2 where the real root r_0 is the first value of *u* with the corresponding v = 0.

Observe that the next in sequence is the first complex root *r* which in this example has a positive real component *(u1* - .00417, v = .07579). This is contrary to the commonly held belief that the values of *u* are uniformly negative. Also note that the v_s shown in Table 1 are all positive. In fact, as noted earlier, the same v. appear with a negative sign for the conjugate roots. The results generated by these data may be seen in Table 2 for the same values of *k* on which the previous table was based.

The values of *i* and the corresponding maximum values of the components of the birth trajectory corresponding to each complex root when they appear for the first time are shown in the last two columns. The first value of *C* corresponds to the real root for which *D* = 0 as it should be. This *C* is also the same as Q_0 and as such measures the contribution of the real root towards the birth trajectory at t=0. The contributions of some of these roots individually

TABLE 2 : COMPONENTS OF TOE COEFFICIENTS OF THE TRAJECTORY OF BIRTHS IN A STABLE MODEL

(α = 15)

u	v	*k*	*L*	*M*	*F*	G	*C*	*D*	τ	*0(τ)*
.00801	0	.008	28.737	0	27.695	o	1.038	0	-	-
		.010	29.313	0	27.695	0	1.058	0	-	-
		.012	29.890	0	27.695	0	1.079	0	-	-
,00417	.07579	.008	8.473	-20.481	27.223	-2.745	.383	-.714	14.280	1.720
		.010	8.523	-20.960	27.223	-2.745	.387	-.731	14.360	1.756
		.012	8.573	-21.439	27.223	-2.745	.390	-.748	14.436	1.793
,00734	.15268	.008	-3.955	-6.5S2	25.901	-5.106	-.099	-.274	12.511	.617
		.010	-4.191	-6.634	25.901	-5.106	-.107	-.277	12.654	.630
		.012	-4.427	-6.687	25.901	-5.106	-.116	-.281	12.804	.643
-.02647	.23189	.008	-.938	-5.694	23.988	-6.793	.026	-.230	6.174	.393
		.010	-1.023	-5.699	23.988	-6.793	.023	-.231	6.236	.394
		.012	-1.108	-5.704	23.988	-6.793	.020	-.232	6.297	.394
-.05298	.31486	.008	1.155	1.575	21.835	-7.677	.025	.081	5.772	.211
		.010	1.197	1.743	21.835	-7.677	.024	.088	5.667	.229
		.012	1.238	1.910	21.835	-7.677	.023	.096	5.568	.248
-.08617	.40319	.008	-9.779	-5.513	19.789	-7.812	-.332	-.410	5.371	1.295
		.010	-9.999	-5.600	19.789	-7.812	-.341	-.417	5.383	1.321
		.012	-10.219	-5.687	19.789	-7.812	-.349	-.425	5.388	1.348

as well as jointly may be seen in Figures 1-6 where the trajectory of births is shown over different lengths of time.

For notational simplicity in the figures the following abbreviations have been used for the different components of the birth trajectory (see eqn. 30). These are

$$B_0(t) = Q_0 e^{r_0 t} \quad \text{for the real root } r_0,$$

$$B_j(t) = 2[\, C_j \cos(v_j t) - D_j \sin(v_j t)]\, e^{u_j t}$$

for the complex roots $u_j \pm iv_j$ and

$$_iB(t) = \sum_{j=0}^{i} B_j(t)$$

It is interesting to see how the presence of one complex root with a positive real component decelerates the stabilization process. The process would have been faster if the values of *u* were all negative.

6. Summary

The derivation of the coefficients of the exponential functions of the birth trajectory of the stable model is quite complex from computational point of view when those have to be obtained from the initial age composition. In this paper we have proceeded with the assumption of a simple pattern for the time series generated by the births that can be attributed to the initial population. Assuming that this time series can be described by two intersecting lines with appropriate slopes it has been shown that the coefficients can be expressed in terms of simple functions of these slopes and of other parameters derivable from the distribution of the net maternity rates. Incidentally, the two slopes as they are defined in this model are not independent of one another. In fact, one is completely determined by the other. Although, there is an element of arbitrariness in the choice of one of these slopes, it can be seen that making it equivalent to the intrinsic rate of growth will serve the purpose quite well. An application of the results of this exercise on a hypothetical data set reveals the relative simplicity of this procedure.

7. Acknowledgement

I am indebted to Mr. Mike McMullen, a graduate student in our departme for preparing the tables and figures for this paper.

References

Keyfitz, N. 1968, *Introduction to the Mathematics of Population.* Addison Wesley.

Mitra, S. and Martin L. Levin, 1990, "Complex Roots of Lotka's Integral Equation for a Special Model of Net Maternity Rates," paper prepared for presentation at the 1990 meetings of the Southern Demographic Association in Louisville, Kentucky

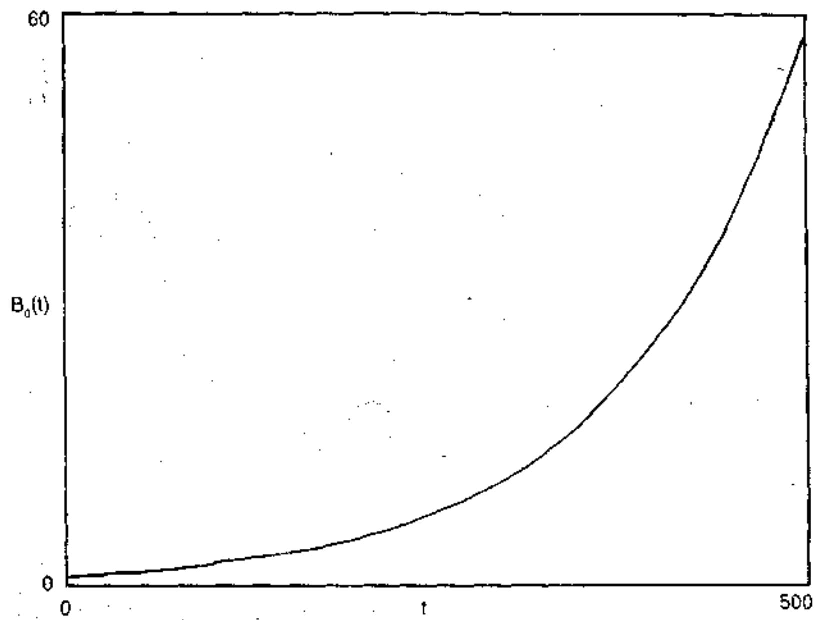

Figure 1. The trajectory of births $B_o(t)$ generated by the real root r_o over a period of 500 years

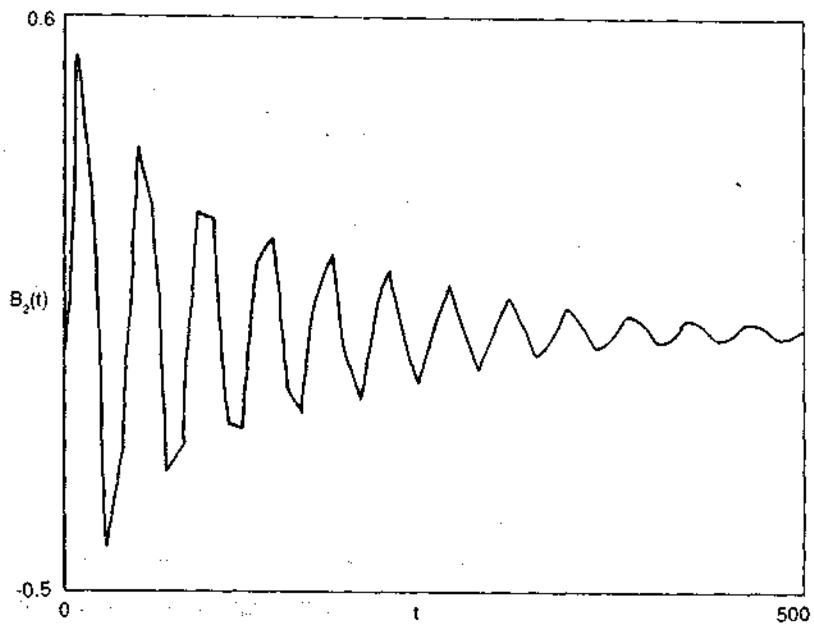

Figure 2. The trajectory of births $B_2(t)$ generated by r_2 and its complex conjugate over a period of 500 years

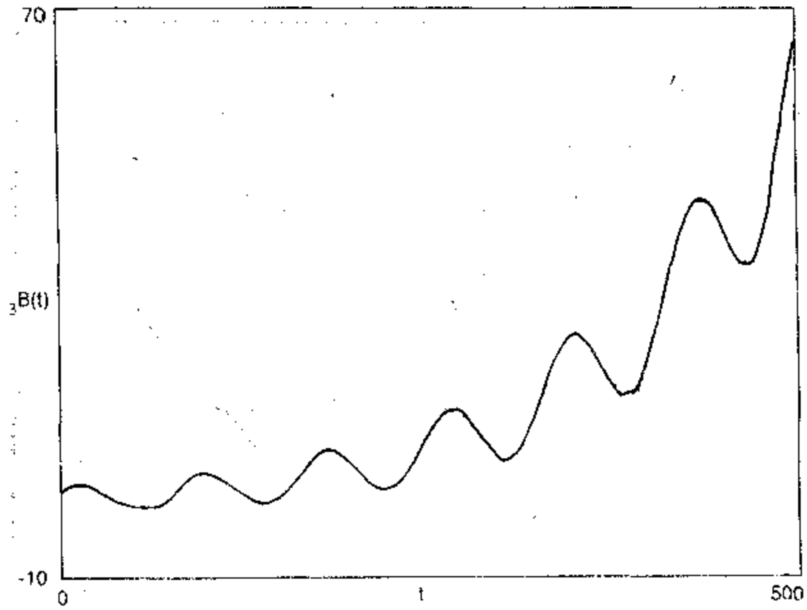

Figure 3. The trajectory of births $_3B(t)$ generated by the real root r_0 and the first three complex roots in decreasing order of u and their conjugates over a period of 500 years

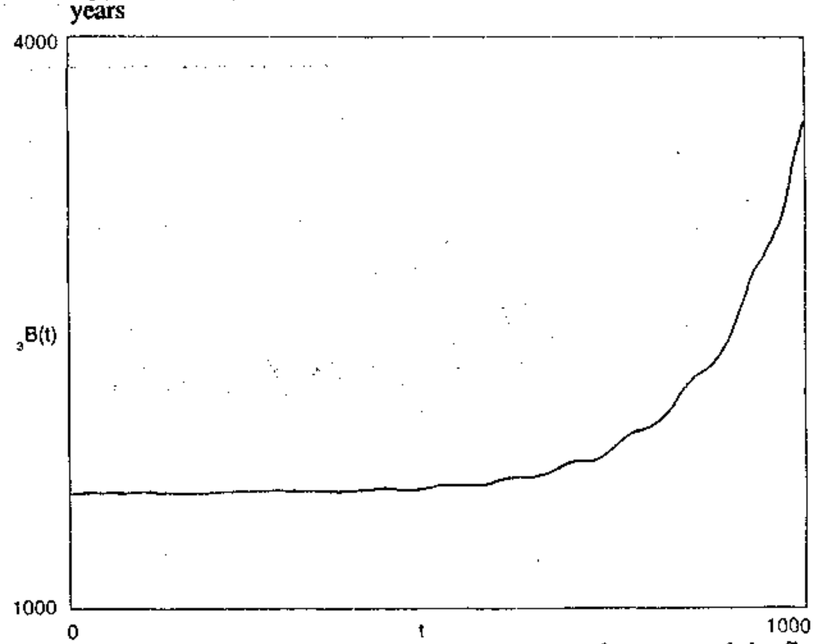

Figure 4. The trajectory of births $_3B(t)$ generated by the real root r_0 and the first three complex roots in decreasing order of u and their conjugates over a period of 1000 years

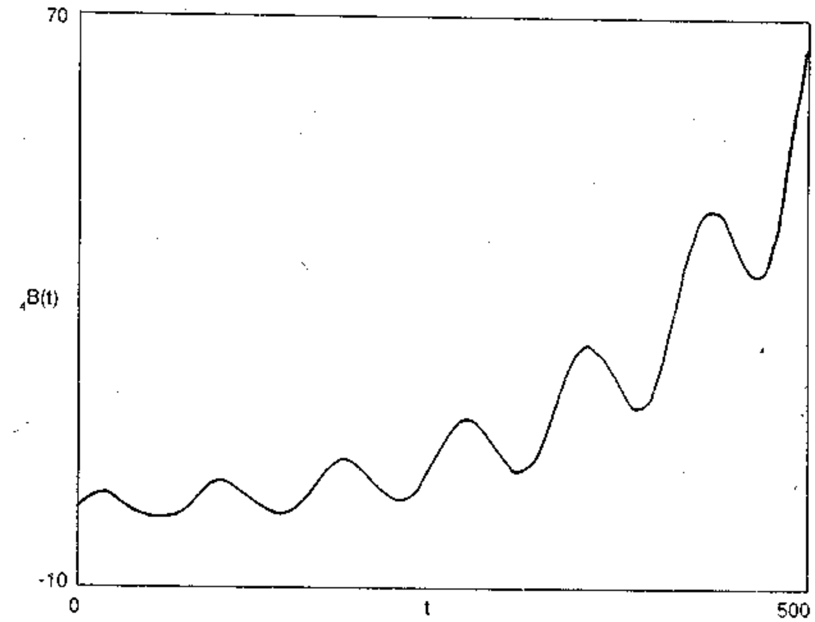

Figure 5. The trajectory of births ${}_4B(t)$ generated by the real root r_0 and the first four complex roots in decreasing order of u and their conjugates over a period of 500 years

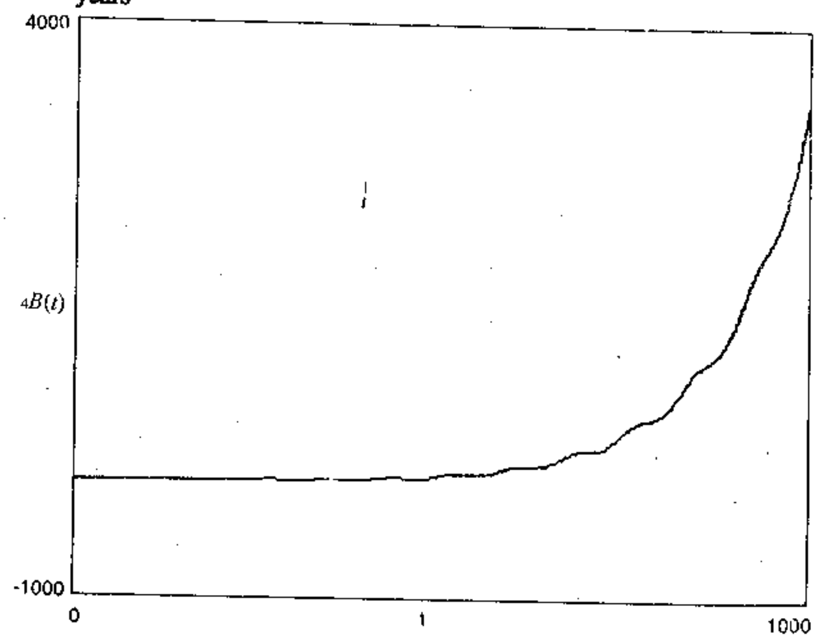

Figure 6. The trajectory of births ${}_4B(t)$ generated by the real root r_0 and the first four complex roots in decreasing order of u and their conjugates over a period of 1000 years