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Population Projection Up toLower Limit of Reproductive Period Under Gradual Change in Fertility Schedula: Some Alternative Models

IN recent years the projections of population under stable and stationary populations have become an important tool for social scientists and planners (Frauenthal 1975; Keyfitz 1971; Mitra 1976; Singh et al. 1981; Yadava 1985). For a closed population, if the age distribution and the future survivorship and maternity behaviour are known, its future age distribution can be worked out. Projection itself is a tedious process and tends towards absurdity because presuppositions may affect the future set-up of the population (Frauenthal 1975). In recent years, however, a number of techniques, although limited in application have been proposed to illustrate the future influence of the current conditions besides several simple and accurate approximations.

If the age-specific birth rate $m(x, t)$ (say) of a stable population drops abruptly to qmx then according to Frauenthal (1975) the birth trajectory is $B(t) = q\ b_o e^{ri}$ for $0 \leq t \leq a$ at time $t = 0$. Decline in mx need not, however, necessarily be uniform (Mitra 1976). This article examines the effect of gradual change in fertility schedule $m(x, t)$ such as $mx\ e^{-rx}$, which also yields a stationary population. Following Mitra (1976) the birth trajectories under gradual change in fertility schedule are obtained under varying situations of maternity function $P(x)\ m(x)$. On the basis of these birth trajectories, the sizes of the future populations up to time $t \leq a$, where a is the lower limit of the reproductive period, are worked out For simplicity, the models deal with the female sex only. The models are also illustrated with some observed data.

Population Sizes at Time t for $t \leq a$

Lotka (1939) has given a renewal equation for female births B(t) at time t for a population which is closed to migration as:

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* The population that is governed by a regime of unchanging fertility and mortality schedules for a long time, with no migration, is called stable population. The age structure of the population remains fixed and the size changes with a constant rate of increase, which depends on fertility and mortality schedules of the population. If the rate of increase becomes zero, the size of the population also becomes fixed. Such a population is known as stationary population.

$$B(t) = \int_t^{\infty} l(x-t) \frac{p(x)}{p(x-t)} m(x, t)\, dx + \int_0^t B(t-x)\, p(x)\, m(x, t)\, dx \qquad (1)$$

for t> 0

where *1 (x)* is the age distribution at time *t* = 0, *p(x)* is the fraction of female population that survives to age *x, m(x) dx* is the probability that a female who is of age *x* will bear a female child in the next dx period of her life.

If the base-line population is stable, we have

$$l(x) = B_0 e^{-rx} p(x) \qquad (2)$$

where B_0 is the instantaneous birth rate at $t = 0$. Thus from equations (1) and (2), we get:

$$B(t) = B_0 e^{rt} \int_t^{\infty} e^{-rx} p(x)\, m(x, t)\, dx$$

$$+ \int_0^t B(t-x)\, p(x)\, m(x, t)\, dx \qquad (3)$$

for $t > 0$

Further for $t \leq \alpha$, equation (3) reduces to:

$$B(t) = B_0 e^{rt} \int_{\alpha}^{\infty} e^{-rx} p(x)\, m(x, t)\, dx \qquad (4)$$

If $P(x)$ remains unchanged and $m(x,t)$ is changed to:

$$m(x,t) = e^{-rx} m(x), \qquad (5)$$

then (4) becomes:

$$B(t) = B_0 e^{rt} \int_{\alpha}^{\infty} e^{-2rx} p(x)\, m(x)\, dx \qquad (6)$$

Below α, the value of $m(x)$ is zero, so (6) can be written as:

$$B(t) = B_0 e^{rt} \int_0^{\infty} e^{-2rx} p(x)\, m(x)\, dx \qquad (7)$$

The integrand in (7) is very difficult to evaluate unless some additional functional form of maternity functionp(x) *m(x)* is assumed. Lotka(1939) has assumed normality and Wicksell's (1931) as incomplete gamma functions of the net maternity functions *p(x) m(x).* According to normality condition (see Mitra 1976) we can wrue:

$$\int_0^{\infty} e^{-2rx} p(x)\, m(x)\, dx = \frac{e^{r^2 \sigma^2}}{R_0} \qquad (8)$$

where *Ro* is the net reproduction rate and o^2 is the variance of age of childbearing. Therefore from (7) and (8), the birth trajectory comes out to be:

$$B(t) = \frac{B_0 e^{rt} e^{r^2 \sigma^2}}{R_0} = \frac{B_0}{R_0} e^{rt + r^2 \sigma^2} \qquad (9)$$

$$\text{for } t < \alpha$$

and according to Wicksell (1931) the value of $m(x)\,p(x)$ can be replaced by:

$$\frac{K C^k x^{k-1} e^{-cx}}{1\bar{k}} \qquad (10)$$

$$x \geq 0$$

Expression (10) is further simplified by Mitra (1976) as:

$$\int_0^\infty e^{-2rx} p(x)\, m(x)\, dx = e^{-r\alpha} \left(\frac{r+c}{2r+c} \right)^k \qquad (11)$$

where

$$c = \mu_1^1 / \sigma^2$$
$$k = \mu_1^2 / \sigma^2$$

where μ_1^1 is obtained by using α as the arbitrary origin so that the average age of childbearing u is equal to $\mu_1^1 + \alpha$. Thus from (11) and (7) the birth trajectory becomes:

$$B(t) = B_0 e^{rt} e^{rt}\; e^{-r\alpha} \left(\frac{r+c}{2r+c} \right)^k \qquad (12)$$

$$\text{for } t \leq \alpha$$

Once the birth trajectory is obtained the population size at any time t *($t \leq \alpha$)* can be obtained following Yadava (1985). The population at any time *t* will be the sum of two population sizes nemely (a) the survivors of birth in (0,t up to time *t* and (b) the survivors of the initial population up to time t. Now let us take the birth trajectory given by expression (9) and the survivors of births in (0,0 up to time t. The total number of births at any time *y*, $0 \leq y < t \leq \alpha$ is

$B_0 e^{ry} \frac{e^{r^2 \sigma 2}}{R_0}$ and the probability that they will survive up to time t is $p(t - y)$. The total survivors up to time t of births in $(0, t)$ are:

$$\int_0^t B_0 e^{ry} \frac{e^{r^2 \sigma 2}}{R_0} p(t-y)\, dy \qquad (13)$$

On putting $(t - y) = z$ in (13), we have:

$$\frac{B_0}{R_0} e^{r^2\sigma^2} e^{rt} \int_0^t e^{-rz} p(z)\, dz \tag{14}$$

Now the population at age y in the base-line population, initially stable, is:

$B_o\, e^{-ry}\, p(a)$ and the probability that they will survive up to time t is $\frac{p(y+t)}{p(y)}$ and hence the total survivors of the initial population at time t will be:

$$\int_0^w B_0\, e^{-ry}\, p(y)\; \frac{p(y+t)}{p}(y)\; dy$$

$$\text{i.e. } \int_0^w B_0\, e^{-ry}\, p(y+t)\; dy \tag{15}$$

where w is the highest age of life. Putting $(y+t) = z$ in (15) we get:

$$B_0\, e^{rt} \int_t^w e^{-rz}\, p(z)\, dz \tag{16}$$

By adding (14) and (16) one can get the population size at time t $(t \leq \alpha)$ as:

$$B_0\, e^{rt} \frac{e^{r^2\sigma 2}}{R_0} \int_0^t e^{-rz}\, p(z)\, dz + B_0\, e^{rt} \int_t^w e^{-rz}\, p(z)\, dz$$

$$= B_0\, e^{rt} \frac{e^{r^2\sigma 2}}{R_0} \int_0^t e^{rz}\, p(z)\, dz + B_0\, e^{rt} \int_0^w e^{-rz}\, p(z)\, dz - B_0\, e^{rt} \int_0^t e^{-rz}\, p(z)\, dz$$

$$= B_0\, e^{rt} \frac{e^{r^2\sigma 2}}{R_0} - 1 \int_0^t e^{-rz}\, p(z)\, dz + e^{rt} \int_0^w B_0\, e^{-rz}\, p(z)\, dz$$

$$= e^{rt} \left(\frac{e^{r^2\sigma 2}}{R_0} - 1 \right) \int_0^t B_0\, e^{-rz}\, p(z)\, dz + e^{rt}$$

$$\text{since} \int_0^w B_0\, e^{-rz}\, p(z)\, dz = 1$$

$$= [1 - (1 - \frac{e^{r^2\sigma 2}}{R_0})\; \bar{A}t\,]\; e^{rt} \quad \text{Model } A_1 \tag{17}$$

where $\bar{A}t = \int_0^t B_0\, e^{-rz}\, p(z)\, dz$ is the proportion of population upto age t in the initial population.

Similarly by using the birth trajectory of expression (12), the population size upto time $t(t < \alpha)$ can be obtained by:

$$[1 - \{1 - e^{-r\alpha}\left(\frac{r+c}{2r+c}\right)^{k}\}\bar{A}\,t\,]\,e^{rt} \qquad \text{Model } A_2 \tag{18}$$

Thus using models A_1 (expression (17) and A_2 (expression (18), the population sizes up to time t for $t \leq \alpha$, the lower limit of the reproductive period, can be calculated. The male population can be obtained from:

$$\text{Male population} = \begin{bmatrix}\text{Female population}\\ \text{X sex ratio at birth}\end{bmatrix} \text{X} \; \frac{\text{Life expectancy for males}}{\text{Life expectancy for females}}$$

Illustrations

The population projection models, obtained by equations (17) and (18) are illustrated for different values of $t < \alpha$ taking numerical values of the parameters involved in the models. For this we need the values of net maternity function *p(x) m(x)* and hence fee values of *r,* the rate of growth, *Ro,* the net reproduction rate, *u,* the mean age of cMdbearing and σ^2, the variance of age of childbearing. To this end, some real data, obtained from the rural areas of eastern Uttar Pradesh, have been taken into account. A survey entitled '.Rural Development and Population Growth' sponsored by the Centre of Population Studies, Banaras Hindu University, was conducted in 1978. The values of *m(x)* for the migrated population (husband migrated and wife living at home permanently), non-migrated population and total population (combining both migrated and non-migrated) were computed by Singh (1985) for these data. The values *of p(x)* are taken from the Regional Model Life Table of Coale and Demeny (1966) (South level 13). This level has been chosen due to the similarity in mortality experience of the region under study. Once the values of *m(x) p(x)* are known, the values *of Ro,* μ and σ^2 can be calculated by the expressions $\mu = R1/Ro$ and

$$\sigma^2 = (R_2/R_0) - (R_1/R_0)^2, \text{ where } R_i = \int_{\alpha}^{\beta} x^i\, m(x)\, p(x)\, dx.$$

The values of μ, σ^2 and R_0 are given in Table 1. Table 2 presents the projected population for different values of $t \leq \alpha$ at various levels of i (growth rate) by model A_1 (given by equation (17). Table 3 shows the size of population for different values of $t \leq \alpha$ at various levels of r and α, by model A_2 (given by equation (18). The initial population is assumed to be 1. For example if we take $r = 0.026$, $R_0 = 2.12$ and $\bar{A}t = 0.1685$ for $t = 5$ (from the Regional Model Life Table of Coale and Demeny, 1966, south level 13). Then the population size after five years would be 1.041 times the initial population (see Table 2). Population is also projected at two levels of R_0 by model A_1. Similarly, if we take $\alpha = 5$, then the values of c and k become 0.45785 and 10.96234 respectively and the size of population after five years by model A_2 becomes 1.042 times of the initial population. At different levels of α and r, the population sizes are computed for $t \leq \alpha$ and are given in Table 3. For different values of r and R_0, the different values of $\bar{A}t$ are taken from the Regional Model Life Table of Coale and Demeny (1966). It should also be noted that the models are illustrated with some numerical values of r *and Ro* which are not necessarily taken from Table 1. It is seen that a has not much impact on the size of projected population for different values of $t \leq \alpha$. One limitation of model Ai seems, however, that with a decrease in the value of *Ro* the size of population increases. It may be due to a smaller value of o^2.

TABLE 1: CALCULATION OF R_0, μ, AND σ^2 WITH THE HELP OF MATERNITY FUNCTION *p(x) m(x)*

		p(a) m(a)		
Age	*p(a)*	*Migrated*	*Non-migrated*	*Total*
15	0.75517	0.132999	0.251369	0.244719
20	0.74104	0.403261	0.550369	0.536520
25	0.72334	0.412441	0.510562	0.500584
30	0.70420	0.416575	0.425007	0.423321
35	0.68450	0.199115	0.259715	0.255387
40	0.60350	0.105650	0.131720	0.130348
45	0.64092	0.071931	0.046241	0.047526
R*o*		1.741967	2.17498	2.128404
μ		29.9615	28,8801	28.9429
r		0.0185	0.0269	0.0263
σ^2		32.9731	55.6645	52.2938

TABLE 2 : POPULATION SIZES FOR DIFFERENT VALUES OF *r* AND Ro BY MODEL Al

	Ro = 2.12, O^2 = 522938			*Ro = 1.75, 0^2 = 32.9731*		
Time (t)	*r*			*r*		
	0.026	*0.020*	0.075	*0.026*	*0.020*	0.075
5	1.04070	1.02211	1.00560	1.05906	1.03771	1.01920
10	1.09427	1.05224	1.01669	1.13217	1.08401	1.04401
15	1.15722	1.08699	1.02975	1.21703	1.13636	1.07165

* Assuming initial population to be 1.

TABLE 3 : POPULATION SEES* FOR DIFFERENT VALUES OF r AND a BY MODEL *A2*

Time (t)	α = 5,c ***= 0.45785, k = 10.96234***			α ***= 10, c = 036224, k = 6.86187***			α ***= 15, c - 026663, k = 3.71754***		
	r			***r***			***r***		
	0.026	***0.020***	0.015	***0.026***	***0.020***	***0.015***	***0.026***	***0020***	***0.01$***
5	1.041866	1.037426	1.030697	1.041771	1.037381	1.030678	1.041615	1.037308	1.030646
10	1.096662	1.083435	1.067087	1.096463	1.083345	1.067049	1.096142	1.083196	1.066985
15	1.160994	1.135465	1.107039	1.160682	1.135325	1.106980	1.160174	1.135094	1.106882

* Assuming initial population to be 1.

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