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A Modified Stochastic Model of Family Formation

Introduction

DEMOGRAPHIC characteristics of women in the reproductive ages are the most important variables taken into consideration in the formulation of population management programmes and working out implementation strategies. Knowledge about ages of mothers at the time of birth of offsprings to them will help health workers to identify the right group of women who should be motivated and persuaded to adopt small family norm as a healthcare and welfare measure. One way of assessing such characteristics of mothers in a population is to go for demographic surveys, which are time consuming and costly affairs. Due to lack of complete registration of births and deaths and incomplete list of eligible couples, planning has been handicapped.

The remedy to this seemingly challenging problem is not hard to find if we realise the underlying mechanism of interaction of nuptiality, fertility and mortality in framing the demographic characteristics of women. With this view in mind, the present paper aims at proposing a modified model to ascertain demographic characteristics of mothers such as average age of mothers at the time of birth of the first and the last offsprings, average reproductive span and probability distribution of birth in the lifetime of mothers with limited demographic information that summarizes the marriage, the fertility and the mortality patterns in a population. The emphasis is to study the sensitivity of nuptiality, fertility and mortality on the aforesaid demographic characteristics of women.

Krishnamoorthy (1979) initiated such investigation using age specific fertility and mortality rates. Similar studies are found in Ram and Pathak (1989), Ladusingh (1991) and Pathak and Guru (1992). However, none of these studies has incorporated age patterns of marriage. To bridge this gap an attempt is made in this paper to develop a modified stochastic model for studying demographic characteristics of mothers in terms of probability

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distribution of number of births, average age of mothers at the time of births of the first and the last offsprings, average reproductive span and probability of ever becoming mother by incorporating age patterns of marriage.

The Model

Let $P_r(y, x)$ denote the probability that a girl will survive and give birth to r children when she reaches age x given that she got married at age $y < x$. The expression for $P_r(y, x)$ is derived under the following assumptions:

(i) the probability that a woman of age x will give birth to a child in the age interval $(x, x + dx)$ is $m(x)\,dx + 0(dx)$, where $m(x) = 0$ outside the reproductive age group,

(ii) the probability that a woman of age x will die in the age interval $(x, x + dx)$ is $\mu(x)\,dx + 0(dx)$ and

(iii) the probability that a woman of age x will survive the age interval $(x, x + dx)$ without giving birth to any child is $1 - \{m(x) + \mu(x)\}\,dx + 0(dx)$.

Under these assumptions, we have

$$P_r(y, x + dx) = P_r^{(y, x)}[1 - \{m(x) + \mu(x)\}\,dx] + P_{r-1}(y, x)\,m(x)\,dx + 0(dx) \quad (1)$$

At the time of marriage of a girl we assumed that

$$P_0(y, y) = 1 \quad \text{and} \quad P_r(y, y) = 0, \quad r > 0 \quad (2)$$

Transposing and taking limit as $dx \to 0$, we get from eqn. (1)

$$P_r(y, x) = -\{m(x) + \mu(x)\}P_r(y, x) + m(x)\,P_{r-1}(y, x) \quad (3)$$

Let $G(r, s)$ be the probability generating function of $P_r(y, x)$, then

$$\frac{\partial}{\partial x} G(r, s) = \sum_{y=0} s^r \frac{\partial}{\partial x} P_r(y, x)$$

$$= -\{m(x) + \mu(x)\}\,G(r, s) + sm(x)\,G(r, s)$$

$$\frac{\partial}{\partial x} \ln G(r, s) = -\{m(x) + \mu(x)\} + s\,m(x)$$

$$\text{or,} \quad G(r, s) = C\,e^{\int_y^x [-\{m(x) + \mu(x)\} + s\,m(x)]\,dt}$$

$$= C_x\,(l_x/l_y)\,e^{-\int_y^x m(t)\,dt}[1 + \ldots + s^r\{\int_y^x m(t)\,dt\}^r \underline{|r} + \ldots] \quad (4)$$

where $l_x = e^{-\int_0^x m(t)\,dt}$ and C is the constant of integration. Using boundary conditions from eqn. (2) in eqn. (4), we obtain $C = 1$, then

$$P_r(y, x) = (l_x/l_y)\, e^{-\int_y^x m(t)\,dt}\; \{\int_y^x m(t)\,dt\}^r / \lfloor\underline{r} \tag{5}$$
$$r = 0, 1, 2, 3, \ldots$$

Let α denote the minimum age eligible for marriage of a girl and the probability that a woman marries in the age interval $(x, x + dx)$ is $g_0(x)dx$ and it vanishes for $x < \alpha$ and $x \geq 50$. Further, let $P_r(\alpha, x)$ be the probability that a woman gives birth to r children when she reaches age x, then

$$P_r(\alpha, x) = (l_x/l_y) \int (l_y/l_0)\, g_0(y)\, e^{-\int_y^x m(t)\,dt}\; \{\int m(t)\,dt\}^r / \lfloor\underline{r}\; dy$$

$$= l_x [\int_\alpha^x g_0(y)\, e^{-\int_y^x m(t)\,dt}\; \{\int_y^x m(t)\,dt\}^r / \lfloor\underline{r}\; dy] \tag{6}$$

Therefore, the probability that a girl will eventually give birth to r children in her lifetime denoted by $P_r(\alpha, .)$ is obtain as

$$P_r(\alpha, .) = \int_0^\omega l_x\, \mu(x)\, [\int_\alpha^x g_0(y)\, e^{-\int_y^x m(t)\,dt}\; \{\int_y^x m(t)\,dt\}^r / \lfloor\underline{r}\,!\; dy]dx \tag{7}$$
$$r = 0, 1, 2, 3, \ldots$$

Now putting $r = 0$ in eqn. (5), we get

$$P_0(y, x) = (l_x/l_y)\, e^{-\int_y^x m(t)\,dt} \tag{8}$$

as the probability of surviving upto age x without giving birth to any child given that the woman was married at age y.

Therefore, the probability that a woman will ever become mother $P(1+)$, is the same as the probability that a woman will eventually have a child in her lifetime after marriage. Thus, we get

$$P(1+) = \int_x^\beta l_x\, m(x)\, [\int_\alpha^x g_0(y)\, e^{-\int_y^x m(t)\,dt}\, dy]dx$$
$$= 1 - P_0(\alpha, .) \tag{9}$$

where β is the upper limit of childbearing.

The mean age of mothers at the birth of first child is obtain as

$$\bar{M}_1 = \frac{\int_x^\beta x\, l_x\, m(x)\, [\int_\alpha^x g_0(y)\, e^{-\int_y^x m(t)\,dt}\, dy]dx}{\int_x^\beta l_x\, m(x)\, [\int_\alpha^x g_0(y)\, e^{-\int_y^x m(t)\,dt}\, dy]dx} \tag{10}$$

At length, the mean age of mothers at the birth of the last child is considered irrespective of her birth history. For this, we have the probability that a woman after marriage at age y gives birth to the last child in the age interval $(x, x + dx)$

$$= (l_y/l_y)\, m(x) \,[\int_x^{x} l_y g_0(y)\, dy]\, e^{-\int_x^{w} m(t)\, dt}\, dx$$
$$= l_x\, m(x) \,[\int_x^{x} g_0(y)\, dy]\, e^{-\int_x^{w} m(t)\, dt}\, dx \qquad (11)$$

As a result the mean age of mothers at the birth of the last child is obtained as

$$\bar{M}_{LB} = \frac{\int_0^{w} x\, l_x\, m(x)\, [\int_x^{x} g_0(y)\, dy]\, e^{-\int_x^{w} m(t)\, dt}\, dx}{\int_0^{w} l_x\, m(x)\, [\int_x^{x} g_0(y)\, dy]\, e^{-\int_x^{w} m(t)\, dt}\, dx} \qquad (12)$$

The average reproductive span (*ARS*) of mothers is evidently

$$ARS = \bar{M}_{LB} - \bar{M}_1 \qquad (13)$$

The formulation in this section is interesting from the point of view of family building strategies based on alternating marriage pattern, because these formulae are expressed in terms of arbitrary marriage pattern depicted by probability density function $g_0(x)$. The presence of $g_0(*)$ makes the present conceptualisation distinct from those of Krishnamoorthy (1979) and other existing works.

Selection of $g_0(x)$

The empirical evidence on the age patterns of marriage suggests a negative skew distribution which gradually increases from nearly zero to a peak value and then steeply tappers to zero again with the advancement of age. Such distributions are closely represented by lognormal distribution. Apart from simplicity in application due to its alliance with, the normal distribution, it also belongs to double exponential family. Variations in the age patterns of marriage can be imparted through the two parameters of lognormal distribution. With these considerations $g_0(x)$ is taken as a lognormal distribution in the present development. Before implementation, a brief discussion on the properties of $g_Q(x)$ as a p.d.f. describing age patterns of marriage is taken up.

Let Y be a r.v. representing female age at marriage in a birth cohort and as above a is the minimum value of Y then c.d.f. $G_Q(y)$ is given by

$$\begin{aligned} G_0(y) &= P[Y \leq y] \\ &= \int_{\alpha}^{y} g_0(s)\, ds \\ &= \Phi\, \{(\ln (y - \alpha) - \mu)/\sigma\} \end{aligned} \qquad (14)$$

for $\alpha < y$, where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2} t^2}\, dt$

Illustration of The Model

The model formulated in Section 2 by incorporating age patterns of marriage is illustrated for studying demographic characteristics of mothers in terms of probability of ever becoming mother, mean ages at the time of births of the first and last offsprings, average reproductive span and net fertility rate.

The main objective of such an illustration is not to claim the goodness of fit of the proposed model but to explore the underlying mechanism of the interplay of nuptiality, fertility and mortality in moulding the aforesaid demographic characteristics of mothers. Keeping this in view the populations -*of* Madhya Pradesh, Kerala and all India with total marital fertility rates of 5.04 children, 4.03 children and 4.32 children per mother on an average, with life expectancy of 51.5 years, 69.9 years and 56.4 years respectively in 1981 are considered for illustration. The median age at marriage of females are respectively 16.5 years, 18.9 years and F7.4 years in rural Madhya Pradesh, Kerala and all India circa 1981 data. The age patterns of marriage are generated through lognormal distribution discussed in Section 3, estimating the two parameters μ and σ by moments method making use of age distribution of marriage in 1981 available from RG (1988) and α is taken as 15 years. The inputs are available in five years age group while the model formulated is continuous in time. Therefore the model is discretised.

The mean ages of mothers at the time of births of the first and the last offsprings are important characteristics of mothers as they provide fertility index in terms of average number of years mothers spent in childbearing. Table 1 shows the mean age of mothers at the birth of first child ($\overline{M}_1$) at the birth of the last" child ($\overline{M}_{1B}$) and average reproductive span (ARS) alongwith total marital fertility rates (TMFR) by regions for the three populations under discussion.

TABLE I: AGE CHARACTERISTICS AND REPRODUCTIVE SPAN OF MOTHERS

Country/State	*Rural*				*Urban*			
	TMFR	($\overline{M}_1$)	($\overline{M}_{1B}$)	*ARS*	*TMFR*	($\overline{M}_1$)	($\overline{M}_{1B}$)	*ARS*
All India	4.50	21.3	36.8	15.5	3.76	20.9	33.5	12.6
Kerala	4.10	20.4	32.8	12.4	3.84	20.2	31.1	10.9
Madhya Pradesh	5.21	20.8	37.6	16.8	4.40	20.6	34.4	13.8

A look at the figures of $\overline{M}_l$ and $\overline{M}_{1B}$ reveals that the mean age of mothers at the time of births of the first and the last children depends on the age patterns of childbearing and significantly on the mortality patterns. Consequently the same conclusion holds good for the average number of years spent in childbearing. This is evident from the fact that Madhya Pradesh with the highest TMFR and the least $e_0°$ has the highest ARS both in rural and urban areas while in Kerala with lowest TMFR and highest $e_0°$ has the least ARS.

In Madhya Pradesh mothers spent almost 17 years and. 14 years respectively in childbearing process in rural and urban areas while in Kerala the corresponding figures are 12 years and 11 years respectively. The marginal difference in the mean age at the birth of first child in rural areas of Kerala and Madhya Pradesh is due to comparatively higher values of age specific marital fertility rates in the first two five years reproductive age group in Kerala. It is noticed that in rural areas where TMFR is quite high, mothers continue childbearing upto late-thirties while in urban areas where TMFR is comparatively low, mothers are relieved from the burden of childbearing in early-thirties. Thus, in brief we conclude that the age characteristics of mothers at the births of the first and the last children are very much dominantly shaped by the age patterns of childbearing.

To add to our understanding of the underlying mechanism of the interplay of nuptiality, fertility and mortality in shapaing demographic characteristics of mothers, in Table 2 below the age patterns of marriage probabilities generated through lognormal distribution are shown for the three populations under consideration.

It can be seen that the probability of getting married in the first quinquennial age group 15-19 years is the highest in all the three populations and majority of females got married by 25 years of age. In rural areas females are more likely to get married at younger ages than in urban areas, more so in Madhya Pradesh than that in Kerala and all India. It can also be noted that marriage pattern in Kerala has quite distinctive feature with only 69 percent and 61 percent of women getting married in 15-19 years age group in rural and urban areas respectively, while the corresponding proportions of females in rural and urban areas of Madhya Pradesh are about 90 percent and 82 percent respectively. In Kerala, a sizeable proportion of females are married after 25 years of age while in Madhya Pradesh particularly in rural areas negligible proportion of females are unlikely to remain unmarried after 25 years of age. For all India, the median age at marriage of females in rural and urban areas are 16.0 years and 17.4 years which are intermediate between the figures of Kerala and Madhya Pradesh. The age patterns of marriage in Madhya Pradesh and all India are more negatively skewed as compared to that of Kerala.

TABLE 2: AGE PATTERNS OF MARRIAGE PROBABILITIES GENERATED FROM LOGNORMAL DISTRIBUTION

Age Groups	*All India*		*Kerala*		*Madhya Pradesh*	
	Rural	*Urban*	*Rural*	*Urban*	*Rural*	*Urban*
15-19	0.8621	0.8203	06915	0.6115	0.9032	0.8186
20-24	0.1200	0.1516	0.2600	0.2296	0.0869	0.1602
25-29	0.0143	0.0222	0.0389	0.0838	0.0082	0.0175
30-34	0.0027	0.0044	0.0072	0.0374	0.0013	0.0029
35-39	0.0006	0.0100	0.0017	0.0197	0.0004	0.0008
40-44	0.0003	0.0005	0.0007	0.0112	0.0000	0.0000
45-49	0.0000	0.0000	0.0000	0.0068	0.0000	0.0000

The probability of ever becoming mother *P(I+)* in the lifetime of a girl is the other demographic characteristic of mothers included in.the present analysis alongwith the net fertility rate (NFR). Net fertility rate is concerned with average number of children a girl child just born would bear in her lifetime when subjected to the prevailing age patterns of nuptiality, fertility and mortality. The probability of ever becoming mother and net fertility rate are in fact inbedded in the probability distribution of number of births. Table 3 shows the probability distribution of number of births a girl child just born would bear in her lifetime. The values of *P(l+)* and NFR are also included.

Scanning at the values of P(/+), it is observed that the probability of ever becoming mother is the highest for Kerala with the lowest TMFR and highest e°and it is the least for Madhya Pradesh with the highest TMFR and the least e°_0 For all India with intermediate TMFR and e°, the value of *P(I+)* lies between the corresponding figures of Kerala and Madhya Pradesh. From such revelation, it may be concluded that the probability of ever becoming mother is influenced more dominantly by age patterns of mortality over age patterns of childbearing. In Kerala where the life expectancy is quite high, as nearly as 91 percent of the girls just born are likely to become mothers in their lifetime when subjected to the prevailing age patterns of marriage, childbearing and mortality circa 1981 as compared to a corresponding figure of nearly 70 percent in Madhya Pradesh and 81 percent in all India level. This seemingly puzzling paradox is explained by the nature of high mortality in Madhya Pradesh which eliminates females during childbearing age groups. The dominant effect of mortality is also noticeable from the probability distributions of number of births, as the death of mother competes with childbearing. These probability distributions also reveals that probability of having higher order births are higher for populations with higher values of TMFR.

TABLE 3: PROBABILITY DISTRIBUTION OF NUMBER OF BIRTHS IN THE LIFETIME OF AN EGO

No of Births	*Alt India*		*Kerala*		*Madhya Pradesh*	
	Urban	*Rural*	*Urban*	*Rural*	*Urban*	*Rural*
0	0.1819	0.1907	0.0857	0.0900	0.2931	0.2957
1	0.0766	0.0476	0.0769	0.0636	0.0448	0.0305
2	0.1360	0.0959	0.1455	0.1279	0.0882	0.0625
3	0.1667	0.1380	0.1850	0.1735	0.1264	0.0998
4	0.1548	0.1520	0,1767	0.1773	0.1396	0.1250
5	0.1156	0.1350	0.1353	0.1451	0.1250	0.1274
6+	0.1684	0.2408	0.1949	0.2226	0.1829	0.2591
NFR	3.10	3.45	3.49	3.63	2.99	3.30
σ(NFR)	2.14	2.27	1.89	1.92	2.43	2.60
P(1+)	0.8181	0.8093	0.9143	0.9100	0.7069	0.7043

The net fertility rate (NFR) introduced in the present analysis is an indicator of childbearing potential under the interaction mechanism of nuptiality, fertility and mortality. Mothers in the state of Kerala which has lower TMFR, higher median age at marriage and higher $e_0°$, has larger net fertility rate as compared to that of Madhya Pradesh which has higher TMFR, lower median age at marriage and lower $e_0°$. The mothers in rural and urban areas of Kerala have on an average 3.63 children and 3.49 children while the corresponding figures in Madhya Pradesh are 3.30 children and 2.99 children respectively. The figures for all India in rural and urban areas are 3.45 children and 3.10 children per mother on an average. This observation suggest that the net average number of children born in the lifetime of a girl is moulded more predominantly by age patterns of mortality rather than by age patterns of marriage and childbearing as observed in 1981. When we turn to the standard deviation in the number of children born in the lifetime of girls denoted by σ (NFR), it comes to light that the higher the value of TMFR and the lower the value of $e_0°$, the greater is the heterogeneity in the average number of children born in the lifetime of a girl subjected to the prevailing age patterns of marriage, childbearing and mortality.

To make a better understanding of the effect of age patterns of marriage on the characteristics of women under discussion in the present paper, we give figures of these characteristics of women in Table 4, obtained without incorporating age patterns of marriage.

TABLE 4: DEMOGRAPHIC CHARACTERISTICS OF WOMEN'

Demographic	*Country/Stale*		
	All India	*Madhya Pradesh*	*Kerala*
P (1+)	0.8373	0.7584	0.8169
NFR	4.14	3.99	2.45
σ (NFR)	2.97	3.09	1.84
$\underline{M}_1$	21.64	20.72	53.05
M_{1B}	35.64	36.28	32.08
ARS	14.00	15.60	9.03

Based on 1981 data and without incorporating age patterns of marriage.
Source: Tables 1 and 3 in Ram, F. and Pathak. K B. (1989). *op. cit.*

Comparison of the values of $\overline{M}_{1B}$ in Table I and Table 4 reveals that age patterns of marriage does not affect the mean age of mothers at the birth of the last child. A similar comparison of the values of $\overline{M}_1$ however suggest that the mean age of mothers at the birth of the first child not only depends on the age patterns of childbearing but also on the age patterns of marriage. The probability of ever becoming mother has remain more or less same, even when no age patterns of marriage are considered. However, when age patterns of marriage is not incorporated, the net fertility rate seems to be overestimated and the effect of mortality is also not clearly visible. This is due to the fact that in the formulation of Ram and Pathak (1989), a woman's chance of dying after childbirth is not considered. From the present comparison, we observed that the modified formulation proposed in this chapter can display the demographic characteristics of women more vividly and logically.

Conclusion

A modified stochastic model of family formation is proposed in this paper by incorporated age patterns of marriage characterised by a displaced lognormal distribution, under some simplified assumptions. The utility of the proposed model lies in its ability to explain the latent characteristics of reproductive process. Major findings of the present paper are:

(i) The mean age of mothers at the time of births of the first and the last offsprings are sensitive to age patterns of childbearing which is characterised by age specific marital fertility rates,

(ii) The probability of ever becoming mother in the lifetime of a girl depends predominantly on mortality and not convincingly on age patterns of fertility. As a consequence the higher the life expectancy, the higher is the probability of ever becoming mother in the lifetime of a girl,

(iii) The average number of children per mother is dominantly moulded by mortality and level of fertility,

(iv) There is much heterogeneity in the average number of children born in the lifetime of mothers for populations with higher TMFR and lower life expectancy.

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