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A Fertility Model and its Application

Introduction

IN recent years the study of the variation in the number of births to a couple has attracted considerable attention. Besides empirical investigations, probability models are being attempted for understanding the nature of this variation. Brass (1958), Dandekar (1955), Henry (1956), Sheps and Perrin (1966), Singh (1963, 1968), Pathak (1966) and others have used mathematical models for explaining the observed distribution of births under different sets of assumptions about human reproduction. These models provide estimates of fertility parameters (fecundability, sterility, etc.) and are useful for prediction. These models take due account of the fecundity or fertility status of females, none of them deals explicitly with the probability of birth at the end of the observational period.

The paper, accordingly, deals with the derivation of a probability model of the number of births to a female during a given period of t years. The applications of the model are discussed and, as an illustration, it is fitted to observed data.

Model

Suppose that a female is married at time 0 and $X(T, t)$ denotes the number of children born to her during the period [7", $T + t]$ given that she has experienced a birth at time $T + t$, where T is a distant point since marriage or menarche whichever is latter. Obviously, the female is not in the state of permanent sterility at time $T + t$.

Suppose that for $r=1, 2, \ldots$ the r-th birth occurs at time $T_1+T_2 + \ldots + T_r$ where T_1 is the time of first birth since marriage and T_r $(r > 1)$ is the time between the $(r - 1)$-th and r-th birth. Further assume that the intervals $T_1, T_2, \ldots$ are mutually independent random variables and $T_2, T_3, \ldots$ are identically distributed with probability density function (p.d.f.) $f(t)$, but T_1 has a different p.d.f. since the starting point of T_1 is not a birth as in the case of other T_r $(r > 1)$ (Perrin and Sheps, 1964). It follows that the birth process is a modified renewal process (Cox and Miller, 1965). If the births are counted from an arbitrary fixed point, T, which is at a considerable distance from marriage, and if $T_1^* + T_2^* + \ldots + T_r^*$ is the time of occurrence of the r-th birth from T, an equilibrium renewal process is obtained. For the equilibrium renewal process, T_r^* $(r > 1)$ has p.d.f. $f(t)$, same as that of T_r $(r > 1)$ and the p.d.f. of T_1^* is

$$f_1^*(t) = \frac{1 - F(t)}{\mu}, \tag{1}$$

and the corresponding Laplace transform is

$$\frac{1 - \phi(s)}{\mu s}, \tag{2}$$

where $$\mu = E(T_r), \quad r > 1,$$

$F(t)$ and $\phi(t)$ are the distribution function and Laplace transform corresponding to $f(t)$ (Cox and Miller, 1965).

Since $T_1^*, T_2^*, \ldots$ are mutually independent, the Laplace transform of $T_1^* + T_2^* + \ldots + T_r^*$ is

$$\frac{(1 - \phi(s))(\phi(s))^{r-1}}{\mu s} = \frac{(\phi(s))^{r-1}}{\mu s} - \frac{(\phi(s))^r}{\mu s}. \tag{3}$$

The inverse of (3), the p.d.f. of $T_1^* + \ldots + T_r^*$, $f_r^*(t)$ say, is

$$f_r^*(t) = \frac{1}{\mu} \int_0^t f^{(r-1)*}(x)dx - \frac{1}{\mu} \int_0^t f^{r*}(x)dx, \tag{4}$$

where $f^{r*}(t)$ denotes the r-fold convolution of $f(t)$ with itself. In the equilibrium process, the birth rate at time $T + t$, which is independent of time, is $1/\mu$ and hence $P[X(T, t) = r]$, the probability that a female has given exactly r births in $[T, T + t]$ given that the r-th birth occurs at time $T + t$ is

$$P[X(T, t) = r] = \frac{f_r^*(t)}{1/\mu} = \int_0^t f^{(r-1)*}(x)dx - \int_0^t f^{r*}(x)dx, \quad r = 1, 2, \ldots \quad (5)$$

which does not depend on T and henceforth will be denoted by $P[X(t) = r]$.

The assumption, of Brass (1958) and Dandekar (1955), Singh (1968), that a female has (*i*) a constant risk of conception, m, (*ii*) a constant period of non-susceptibility associated with each delivery, h and (iii) one-to-one correspondence between a conception and a live birth, satisies the conditions laid down in this model and it is easy to verify that

$$f(t) = m\, e^{-m(t-h)}, \quad t > h \quad (6)$$

$$f^{r*}(t) = \frac{m^r}{< r}\, e^{-m(t-rh)}(t - rh)^{r-1}, \quad t > rh \quad (7)$$

and

$$\int_0^t f^{r*}(x)dx = 1 - e^{-m(t-rh)} \sum_{s=0}^{r-1} \frac{\{m(t - rh)\}^s}{s!}, \quad t > rh. \quad (8)$$

These expressions when substituted in (5), yield

$$P[X(t) = r; m, h] = e^{-m(t-rh)} \sum_{s=0}^{r-1} \frac{\{m(t - rh)\}^s}{s!},$$

$$-e^{-m(t-\overline{r-1}h)} \sum_{s=0}^{r-2} \frac{\{m(t - \overline{r-1}h)\}^s}{s!} \quad r = 1, 2, \ldots, n - 1 \quad (9)$$

$$= 1 - e^{-m(t-\overline{r-1}h)} \sum_{s=0}^{r-2} \frac{\{m(t - \overline{r-1}h)\}^s}{s!} \quad r = n,$$

where $n = [t/h] + 1$ is the maximum number of births to a female in $[T, T+t]$ and $[t/h]$ stands For the greatest integer not exceeding t/h. It may be noted that if the last birth is not counted, the distribution of $X(t)$ reduces to that of Singh (1968) with t replaced by $t — h$.

The probability expressions could be modified by incorporating the chances of foetal death using the results derived in Singh *et al.* (1973).

Application

For illustration, the model has been applied to the data of the Demographic Survey of Varanasi (Rural) 1969-70. The survey was conducted by Demographic Research Centre, Banaras Hindu University. About 2200 households scattered in *52* villages were selected using a two-stage stratified random sampling procedure. Among other things, the information on pregnancies within last seven years was obtained for each currently married female under 50 years of age. In the sample there were 337 females with marital duration more than 10 years, who had experienced their latest birth in the last year (October 1968 to September 1969). Table 1 presents the observed distribution of these females according to the number of births inclusive of the latest during 6 years preceding the termination of the last birth. The average duration of post-partum amenorrhea (PPA) based on the data for live births in the survey is about 9 months. Further, a mixture of two type III distributions fitted to data on PPA for each year of birth gave a low average of the PPA for about half the females and a high average for the rest (Singh and Bhaduri, 1971).

In view of the above, let us assume that (i) females have the same conception rate, m, and (ii) a and $1 — a$ are the proportions of females with h_1 and h_2 units of durations of non-susceptibility respectively. Now the probability that a female selected at random has exactly r births is :

$$P[X(t) = r] = aP[X(t) = r; h_1] + (1 - a) P[X(t) = r; h_2], \qquad (10)$$

where a is the probability that a female giving birth at any particular point of time in the equilibrium process belongs to the group with $h — h_1$.

In the present case $t = 6$ years. In order to apply the model of expression (9), we take $h = 1.50$ years (9 months of gestation and 9 months of PPA) and for the application of the second model (10), we assume h_1 = 1.25 years, h_2=1.75 years and a = 0.50.

The estimate of m is obtained by equating the observed mean to the corresponding theoretical expressions in both the cases. The expected frequencies and the estimates are given in Table 1.

TABLE 1—DISTRIBUTIONS OF FEMALES ACCORDING TO THE NUMBER OF BIRTHS DURING 6 YEARS PRECEDING THE TERMINATION OF BIRTH IN THE PERIOD OCTOBER 1968 TO SEPTEMBER 1969

Number of births	*Observed number of women*	*Expected number of women*	
		h = 1.5 years	*h_1 — 1.25 years, h_2 = 1.75 years, a = 0.50*
1	27	22.7	22.9
2	133	133.3	135.0
3	148	159.8	151.1
4	29	21.2	28.0
Total	337	337.0	337.0
χ^2(d.f. 2)		4.60	0.85
$\hat{m}$		0.60	0.60

Though both the models describe the data, the one which takes account of the variation in h is a better approximation and is consistent with the bimodal distribution of PPA observed in the survey data.

The model may be used to assess the effectiveness of a family planning programme where couples want to limit the family size just after a certain number of births. The estimates of the parameters of present model obtained from the data on the number of births to females during the preceding t years can be utilized to get an expected distribution. A comparison of the expected and observed distribution of the number of births to a couple during a given period of time after the introduction of the programme provides a measure to evaluate the effect of the programme. The model with suitable modifications may find also its application in the field of morbidity, accident proneness etc.

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