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An Analytical Model for Estimating the Incidence of Secondary Sterility

STERILITY, the biological incapability of a woman to bear a live child, forms an important part of the study of human fertility. A woman who does not experience any pregnancy resulting in a live birth is called primarily sterile, while the infecundity of a woman who did experience such a pregnancy is termed secondary sterility. Knowledge about the proportion of sterile women by age and parity helps the determination of the level of fertility of a population. This parameter commands utility in the estimation of the eligibility of couples for family planning programmes. However, to establish the sterility of a woman is not easy. 'It would necessitate thorough medical check up. Symptoms of sterility remain largely undetected and get eventually confounded with menopause. This causes a serious deficiency in the data about the incidence of secondary sterility, particularly in the developing countries.

In this context, the research workers have suggested indirect procedures to approximate it. Parity progression ratios and the age at last live birth for a woman of completed fertility have been used frequently in this connection. A similar approach is to consider the open birth interval, the interval between the age of the woman at the time of enquiry and her age at last live birth, for a non-contracepting woman as an indicator of her sterility. The open birth interval of a woman of completed fertility is the duration between her age at last live birth and menopause. A long open birth interval in the case of a non-contracepting woman would no doubt, raise a doubt about her fecundity, but this should not necessarily mean complete sterility. A sub-fecund woman May take longer to experience a live birth than the period of observation or the open birth interval. It seems, thus, useful to evolve some methodology where such possibilities are taken into account. The present paper attempts an analytical model, based on a simple probability theory, for estimating the incidence of secondary sterility by utilising the information on parity and open birth interval for a group of married women.

The Model

A woman may become sterile at any time in her reproductive span of life. The incidence of sterility may depend on her age or parity or both. The exact nature of this relationship is yet to be established. It is, however, true that a woman runs an additional risk of becoming sterile at the time of termination of a pregnancy. It has been assumed here, for the sake of simplicity, that the incidence of secondary sterility is a function of a woman's parity only and it can occur to a woman at the time of termination of a pregnancy. Then the probability that she becomes sterile at any other time in her reproductive life (excluding menopause) is zero. This may sound slightly illogical from the biological point of view but makes little difference in the study of her fertility behaviour.

No explicit assumption is made in the model to consider the effect of age on the occurrence of sterility. This can largely be taken into account by apply-iny the model to a group of women of a specific marriage cohort, as age and parity are highly correlated for women of same marriage cohort. The model is based on the following further assumptions :

(*i*) The fecundability, or the monthly chance of conception of a married fecund woman of parity i is p_i, and $q_i = 1 — p_i$.

(ii) The probability that a woman will be primarily sterile is α_0.

(iii) The probability that a woman who is initially fecund will become sterile soon after the i-th birth is α_i.

(iv) All conceptions terminate in a live birth.

(v) The period of gestation preceding a live birth is equal to y months. *(vi)* The period of post-partum amenorrhea (PPA) following a live birth is equal to 'm' months, and $h = m + g + 1$. (Generally, PPA is found to have larger variability among women than what is observed in the case of gestation period. However, substituting a mean value for this in the model is not likely to have any appreciable impact on the distribution of parity and open birth interval. Its effect can be reduced by suitably grouping the open birth intervals instead of considering them by single months).

(vii) A woman does not use any method of family planning during the period for which she has been observed.

Given these assumptions an attempt is made to derive an expression for the probability that a women having 't' months as her marital duration will experience i births and have an open interval of x months. Let the probability be denoted by $P^t_{i,x}$. To determine $P^t_{i,x}$, it is essential to consider the occurrence of the following two independent events :

A. the woman should remain fecund till she has given i births, and the z'-th birth should occur at a distance of x months from the point of observation, and

B. the woman may or may not remain fecund after the occurrence of the i-th birth, but she should not have an additional birth within t months so that her open interval of x months is maintained. In fact, this event can be further classified into the following three mutually exclusive and exhaustive cases :

(i) the woman becomes sterile after the z'-th birth,

(ii) the woman is still fecund but could not have any further conception due to lack of sufficient waiting time, and

(iii) the woman is fecund, had another conception and is in thej-th ($j = 1, 2, \ldots, g - 1$) month of pregnancy at the time of observation.

The event *A* becomes irrelevant for evaluating $P^t_{0,t}$ and we are left to consider the three mutually exclusive and exhaustive events as mentioned in *B :*

$$P^t_{0,t} = \alpha_0 + (1 - \alpha_0)\, q_0^t + (1 - \alpha_0) \sum_{j=1}^{g} q_0^{t-j}\, p_0. \tag{1}$$

For $P^t_{i,x}$ when $x \leqslant m,$ the consideration of the three events of *B* is not required as the woman who is still in the post-partum nonsusceptible period following the *i-ih* birth does not have any chance of a further conception. As the i-th birth in this case should necessarily occur at the *(t — x)-th* month, or in other words the z'-th conception should occur at the *(t — x — g —* l)-th month, the total number of susceptible months in which the woman does not conceive are t - *g* - 1 - (*i* - 1) *h — x* = *T* (say). These *T* months can be distributed in different possible combinations before the *i* conceptions; consequently

$$P^t_{i,x} = (1-\alpha_0)(1-\alpha_1)\ldots(1-\alpha_{i-1})\, p_0 p_1 \ldots p_{i-1} \{q^T_{i-1} + q^{T-1}_{i-1} \sum_{n_1=0}^{i-2} q_{n_1} + $$
$$+ q^{T-2}_{i-1} \sum_{n_1=0}^{i-2} \sum_{n_2=n_1}^{i-2} q_{n_1} q_{n_2} + q^{T-3}_{i-1} \sum_{n_1=0}^{i-2} \sum_{n_2=n_1}^{i-2} \sum_{n_3=n_2}^{i-2} q_{n_1} q_{n_2} q_{n_3} + \ldots +$$
$$+ q^0_{i-1} \sum_{n_1=0}^{i-2} \sum_{n_2=n_1}^{i-2} \ldots \sum_{n_T=n_{T-1}}^{i-2} q_{n_1} q_{n_2} \ldots q_{n_T}\}, \qquad (2)$$

for any $x \leqslant m.$

However, if $x > m$ the mutually exclusive events of *B* should be taken into account for the evaluation of $P^t_{i,x}$, and

$$P^t_{i,x} = F\,(\alpha_i + (1-\alpha_i)\, q_i^{x-m} + (1-\alpha_i) \sum_{j=1}^{g} q_i^{x-m-j}\, p_i$$
$$= F\,(\alpha_i + (1-\alpha_i)\, q_i^{x-h+1} \qquad (3)$$

for $m + 1 \leqslant x \leqslant t - g - 1 - (i-1)\, h,$

where *F* is the right-hand side of the expression given in Equation 2.

It can be easily seen that if we assume fecundability to be constant, that is, substitute $p_0 = p_1 = \ldots = p_{max} = p,$ then Equations 1, 2 and 3 reduce to

$$P^t_{0,t} = \alpha_0 + (1-\alpha_0)\; q^{t-g}, \qquad (4)$$

$$P^t_{i,x} = (1-\alpha_0)(1-\alpha_1)\ldots(1-\alpha_{i-1})\, p^i \; {}^{T+(i-1)}\!\underset{i-1}{C}\; q^T, \qquad (5)$$

for any $x \leqslant m$,

and

$$P^t_{i,x} = (1-\alpha_0)(1-\alpha_1)\ldots(1-\alpha_{i-1})\, p_i \, {}^{T+(i-1)}C_{i-1}\, q^T\{\alpha_i + (1-\alpha_i)\, q^{x-h+1}\},$$

for $x > m$. (6)

It has been shown in Appendix 1 that

$$P^t_{0,t} + \sum_{i=1}^{Z} \sum_{x=0}^{t-g-1-(i-1)h} P^t_{i\,x} = 1,$$

where $Z = \frac{t-g-1}{h} + 1$ is the maximum number (in integer) of birth that can occur in t months.

Different marital durations for women of specific marriage cohort could be considered in the model. If Mi denotes the probability that a woman (of a specific marriage cohort) will have t months as her marital durations, then

$$P_{i,x} = \sum_t M_t P^t_{i,x}$$

is the probability that the woman will give i births and will have an open birth interval of x months irrespective of any specification as to her marital duration. The distribution of open intervals are rarely given by single months in any survey. The values of P_t, ,, 's, therefore, can be suitably modified to get the probability that a randomly selected woman in a survey will be found to have given i births with an open birth interval between, say, Xj-i to Xj months, and let this probability be defined as P_w

$$P_{ij} = \sum_t M_t \sum_{x=X_{j-1}}^{X_j} P^t_{i,x},$$

where $j = 1, 2 \ldots, L$ will indicate the number of classes in which the open birth intervals are subdivided.

The above model describes the variations in parity and open birth interval for a group of non-contracepting woman. The model can be fruitfully employed to estimate number of fertility parameters of a population such as fecundability by parity, primary sterility and secondary sterility by parity. This can be accomplished if data are available regarding the distribution of women (who did not use any family planning method during the period of observation) according to their parity and open birth intervals and the period of non-susceptibility associated with a live birth.

Estimation

The complicated nature of the model makes it unsuitable for estimating its parameters by the usual maximum likelihood estimation procedure or by the method of moments. BAN estimation procedure, developed by Neyman (1949), can, however, be employed for the purpose. This procedure consists in minimizing the quantity

$$Q = \sum_i \sum_x (N_{i,x} - NP^t_{i,x})^2 / N_{ix}$$

with respect to the parameters which are to be estimated. N_{ix} is the observed number of women who are in the i-th parity and have an open interval of x months, and $N = \sum_i \sum_x N_{ix}$. As $P^t_{i,x}$'s are not linear, it is required that they be linearized about any suitable pilot point, so that

$$\bar{P}^t_{ix}(\theta_1, \theta_2 \ldots \theta_r) = P^t_{ix}(\bar{\theta}_1, \bar{\theta}_2 \ldots \bar{\theta}_r) + \sum_{j=1}^{r} b_{ixj}(\theta_j - \bar{\theta}_j);$$

$P^t_{ix}(\theta_1, \theta_2, \ldots, \theta_r) = P^t_{ix}$ with $\theta_1, \theta_2 \ldots \theta_r$ as the r estimable parameters, and

$$b_{ixj} = \frac{dP^t_{ix}}{d\theta_j} \Big/ \theta_1 = \bar{\theta}_1, \theta_2 = \bar{\theta}_2, \ldots, \theta_r = \bar{\theta}_r,$$

where $\bar{\theta}_1, \bar{\theta}_2 \ldots \bar{\theta}_r$ are the pilot estimates of the respective parameters. The estimates of $\theta_1, \theta_2 \ldots \theta_r$ obtained by minimizing

$$\bar{Q} = \sum_i \sum_x (N_{ix} - N\bar{P}^t_{ix}(\theta_1, \theta_2 \ldots \theta_r))^2 / N_{ix} \tag{7}$$

with respect to $\theta_1, \theta_2 \ldots \theta_r$ will also be BAN estimates.

The partial derivatives of $P^{t}_{i,x}$ with respect to different estimable parameters are given in Appendix 2.

Application of the Model

The concept of open birth interval being of recent origin, the number 'of surveys which collected data on this index are very few. It is all the more difficult to find data regarding the distribution of women according to parity and open interval. To illustrate the application of the model we have taken data from the Greater Bombay Fertility Survey of the International Institute for Population Studies which, collected in 1966 such information from individual married woman in the sample (Rele and Kanitkar (1971))*. To simplify the procedure of estimation, the group of women who got married between the age 15 to 19 and had a marital duration of 10 years were considered. Altogether there were only 148 such women who did not use any method of family planning within the period of observation. As mentioned earlier, the consideration of a specific marriage cohort would enable us to take into account the effect of age on sterility to a great extent. This restriction results in a considerable reduction in the available sample size. The sample size could have been increased by including women of the same cohort having different marital durations. However, as a simple illustration we have restricted our study to the above group of 148 women.

As the intention of the paper is to emphasize the procedure of estimation of secondary sterility by parity, some simplifying assumptions are made about the other estimable parameters of the model. These are not the necessary conditions for estimation of secondary sterility, but they have been incorporated in this illustration for the sake of simplicity. The procedure of estimation in a more general case has been described in Appendix 2.

It has been assumed here that the parameters *m*, *g* and «$_0$ are known. Among these three parameters *g* is the least variable one, and it can be safely assumed to have a value equal to 10 months (including the month of delivery).

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On the other hand, the variable m, which is influenced both by the health conditions of a woman as well as her practice of breast feeding, has a larger variability. In the absence of any information about it in the survey, we assume it to be 11 months, following the study by Potter *et al.* (1965) showing a mean = 10.8 months and variance = 46.09 for the distribution of PPA for a group of Indian women. A comparison between the proportion of women who did not give any live birth and $P^{f}_{0,t}$ of Equation 4 for any reasonable level of fecundability indicated a value of α_0 to be equal to .03. It has also been assumed that the fecundability remains constant over parity. Since we are considering a marital duration of 10 years this assumption is not expected to have significant impact on the estimates of secondary sterility. On the other hand, this not only helps us in simplifying the procedure of estimation but also allows us to have sufficient degrees of freedom for testing the goodness of fit.

With these simplifications the $P^{f}_{i,x}$'s given in Equations 4, 5 and 6 have been utilized for estimating α_i (i = 1, 2 and 3) and q. α_4 could not be estimated as the marital duration considered here is not large enough to yield a valid estimate for it. The pilot estimates of these parameters have been taken arbitrarily as

$$q = .93, \alpha_1 = .06, \alpha_2 = .10 \text{ and } \alpha_3 = \alpha_4 = .14.$$

Selection of the pilot estimates in this manner does not, however, affect the final estimates. This may, however, result in an increase in the number of iterations required to achieve the final estimates. The values of the partial derivatives of $P^{f}_{i,x}$ with respect to the different parameters are obtained next. The estimates can now be improved by solving the four equations derived by minimizing the quantity $\bar{Q}$ (Equation 7) with respect to the four parameters. Final BAN estimates have been obtained as

$$\alpha_1 = .108014, \alpha_2 = .285915, \alpha_3 = .201198 \text{ and } q = .927378.$$

Observed and expected values of N_{ij}'s calculated with the help of these estimates have been shown in Table 1 below. A chi-square test of significance was instituted to see the goodness of fit which yielded a value of $\chi^2 = 11.30$ at 7 degrees of freedom. This indicates that the fit is satisfactory.

TABLE 1—OBSERVED AND EXPECTED VALUES OF $N_{i,\ x_{j-1}}$ TO x_j, i.e., THE NUMBER OF i-th PARITY WOMEN HAVING OPEN BIRTH INTERVAL BETWEEN x_{j-1} AND x_j($N_{i,\ x_{j-1+}}$ MEANS THE OPEN BIRTH INTERVAL IS GREATER THAN OR EQUAL TO x_{j-1})

$N_{i,x_{j-1}}$ *to* x_j	*Observed*	*Expected*	Ni, x_{i-i} *to* x_i	*Observed*	*Expected*
N_0^*	5	4	N_3 *0* to 23	20	23
N_1, 0+	16	17	N_3, 24 to 47	18	21
N_2, 0 to 23	6	4	N_3, 48+	7	4
N_2, 24 to 47	12	10	N_4, 0 to 23	24	27.
N_2, 48 to 7[1]	19	19	Ml, 24+	6	4
N_2, 72+	10	13	N_5,0+	5	2
			Total	148	148

*For these women the open birth interval is 120 months.

Conclusions

Despite some of the stringent assumptions incorporated in the model, a reasonably good fit could be obtained for the observed distribution of parity and open birth interval. It is desirable, for examining the goodness of fit, to have a closer look at the observed data. Being a metropolitan city, the Greater Bombay population consisted of a very heterogeneous group of women. Some peculiarities have been observed in the fertility behaviour of the Zoroastrian women which constituted a fairly large proportion of the sample (Rele and Kanitkar (1974)). The sample size considered here is, therefore, hardly sufficient for a study of this kind. Condensation of the data into fewer cells at some parities was resorted to in order to improve the goodness of fit.

Some of the assumptions embodied in the model concerning the values of certain parameters are unavoidable because of lack of pertinent information in the survey. Eleven months of post partum non-susceptible months following a live birth seems to be little more for an urban group of women. This has, however, helped in compensating for the loss of non-susceptible period associated with a fetal loss conception which was assumed to absent in the model.

The model can be used for estimating various fertility parameters of a population. By considering group of women of different marriage cohorts, one can apply the model independently for each of the cohorts to obtain approximate idea about the proportion of women becoming sterile by parity and age. For example, let us consider a group of women all of whom got married at the age 15. *If* p_a and α_1 are estimated for these women by the help of the model, it would be possible to get an idea about the average at which the first birth occurs. Therefore, α_1 would not only indicate how many women become sterile after *1st* birth but would also denote the age at which this occurs.

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APPENDIX 1

We proceed to prove here that

$$P^t_{0,t} + \sum_{i=1}^{Z} \sum_{x=0}^{t-g-1-(i-1)h} P^t_{i,x} = 1.$$

Let

$$t = (Z-1)h + g + 1 + N.$$

Here N can take a value between 0 and $h-1$. For the sake of simplicity we remove the suffix t and assume that $\alpha_0 = \alpha_1 = \ldots = \alpha_2 = \alpha$ and $p_0 = = p_1 = \ldots = p_2 = p$.

The proof given below, however, remains valid without these assumptions. Now,

$$\sum_{x=0}^{N} P_{Z,x} = (1-\alpha)^Z \left[\overset{Z+N-1}{\underset{Z-1}{C}} q^N p^Z + \overset{Z+N-2}{\underset{Z-1}{C}} q^{N-1} p^Z + \ldots + \overset{Z-1}{\underset{Z-1}{C}} q^0 p^Z \right] =$$

$$= (1-\alpha)^Z \left[- \overset{Z+N-1}{\underset{Z-1}{C}} q^{N+1} + \overset{Z+N-2}{\underset{Z-2}{C}} q^N + \ldots + \overset{Z-2}{\underset{Z-2}{C}} q^0 \right] p^{Z-1}. \qquad (1)$$

When $i = Z-1$, x can take any value between 0 and $h+N$. Let

$$\sum_{x=0}^{h+N} P_{Z-1,x} = P^1_{Z-1,x} + P^2_{Z-1,x},$$

where

$$P^1_{Z-1,x} = \sum_{x=0}^{h-1} P_{Z-1,x} = (1-\alpha)^{Z-1} p^{Z-1} \times$$

$$\times \left[\overset{Z+h+N-Z}{\underset{Z-2}{C}} q^{h+N} + \overset{Z+h+N-3}{\underset{Z-2}{C}} q^{h+N-1} + \ldots + \overset{Z+N-1}{\underset{Z-2}{C}} q^{N+1} \right],$$

and

$$P^2_{Z-1,x} = \sum_{x=h}^{h+N} P_{Z-1,x} =$$

$$= \alpha(1-\alpha)^{Z-1} p^{Z-1} \left[\overset{Z+N-2}{\underset{Z-2}{C}} q^N + \overset{Z+N-3}{\underset{Z-2}{C}} q^{N-1} + \ldots + \overset{Z-2}{\underset{Z-2}{C}} q^0 \right] +$$

$$+ (1-\alpha)^Z p^{Z-1}\left[\overset{Z+N-2}{\underset{Z-2}{C}} + \overset{Z+N-3}{\underset{Z-2}{C}} + \ldots + \overset{Z-2}{\underset{Z-2}{C}}\right] q^{N+1} =$$

$$= \alpha(1-\alpha)^{Z-1} p^{Z-1}\left[\overset{Z+N-2}{\underset{Z-2}{C}} q^N + \overset{Z+N-3}{\underset{Z-2}{C}} q^{N-1} + \ldots + \overset{Z-2}{\underset{Z-2}{C}} q^0\right] +$$

$$+ (1-\alpha)^Z p^{Z-1} \overset{Z+N-1}{\underset{Z-1}{C}} q^{N+1}$$

$$\therefore\ P^2_{Z-1x} + \sum_{x=0}^{N} P_{Z,x} =$$

$$= \alpha(1-\alpha)^{Z-1} p^{Z-1}\left[\overset{Z+N-2}{\underset{Z-2}{C}} q^N + \overset{Z+N-3}{\underset{Z-2}{C}} q^{N-1} + \ldots + \overset{Z-2}{\underset{Z-2}{C}} q^0\right] +$$

$$+ (1-\alpha)^Z p^{Z-1}\left[\overset{Z+N-2}{\underset{Z-2}{C}} q^N + \overset{Z+N-3}{\underset{Z-2}{C}} q^{N-1} + \ldots + \overset{Z-2}{\underset{Z-2}{C}} q^0\right] =$$

$$= (1-\alpha)^{Z-1} p^{Z-1}\left[\overset{Z+N-2}{\underset{Z-2}{C}} q^N + \overset{Z+N-3}{\underset{Z-2}{C}} q^{N-1} + \ldots + \overset{Z-2}{\underset{Z-2}{C}} q^0\right]$$

$$\therefore\ \sum_{x=0}^{h-1} P_{Z,x} + \sum_{x=0}^{h+N} P_{Z-1,x} =$$

$$= (1-\alpha)^{Z-1}\left[\overset{Z+h+N-2}{\underset{Z-2}{C}} q^{h+N} + \overset{Z+h+N-3}{\underset{Z-2}{C}} q^{h+N-1} + \ldots + \overset{Z+N+1}{\underset{Z-2}{C}} q^{N+1}\right.$$

$$\left. + \overset{Z+N-2}{\underset{Z-2}{C}} q^N + \ldots + \overset{Z-2}{\underset{Z-2}{C}} q^0\right] p^{Z-1},$$

adding in the similar way the value of $P_{Z-2,x}$ to the above equation it can be shown that

$$\sum_{x=0}^{N} P_{Z,x} + \sum_{x=0}^{h+N} P_{Z-1,x} + \sum_{x=0}^{2h+N} P_{Z-2,x} =$$

$$= (1-\alpha)^{Z-2}\left[\overset{Z+2h+N-3}{\underset{Z-3}{C}} q^{2h+N} + \overset{Z+2h+N-4}{\underset{Z-3}{C}} q^{2h+N-1} + \ldots + q^0\right] p^{Z-2}.$$

Proceeding in the similar way

$$\sum_{i=1}^{Z} \sum_{x=0}^{t-g-1-(i-1)h} P_{i,x} =$$

$$= (1 - \alpha)\, p \left[\overset{(Z-1)h+N}{\underset{0}{C}} q^{(Z-1)h+N} + \overset{(Z-1)h+N-1}{\underset{0}{C}} q^{(Z-1)h+N-1} + \ldots + q^{0} \right] =$$

$$= (1 - \alpha)\, p\, [q^{t-g-1} + q^{t-g-2} + \ldots + 1];$$

adding to these the value of $P_{0,t}$, we have

$$P_{0,t} + \Sigma\, \Sigma\, P_{i,x} = \alpha + (1 - \alpha)\, \{q^{t} + q^{t-1} p + q^{t-2} p + \ldots + 1\} = 1.$$

APPENDIX 2

Estimation Procedure

The values of $dP_{i,x}/dq_j$ and $dP_{i,x}/d\alpha_j$ for $j = 0, 1 \ldots Z$ can be obtained as follows :

$$\frac{\partial P^t_{0,t}}{\partial q_j} = 0, \text{ if } j > 0$$

$$= (1 - \alpha_0)(t - g)\, q_0^{t-g-1}, \text{ if } j = 0,$$

$$\frac{\partial P^t_{0,t}}{\partial \alpha_j} = 0, \text{ if } j > 0$$

$$= 1 - q_0^{t-g}, \text{ if } j = 0;$$

when $x \leqslant m$

$$\frac{\partial P^t_{i,x}}{\partial q_i} = A \Bigg[q_{i-1}^{T-1} + q_{i-1}^{T-2} \sum_{n_1 \neq j = 0}^{i-2} q_{n_1} + q_{i-1}^{T-3} \sum_{n_1 \neq j = 0}^{i-2} \sum_{n_2 \neq j = n_1}^{i-2} q_{n_1} q_{n_2} + \ldots +$$

$$+ q_{i-1}^{0} \sum_{n_1 \neq j = 0}^{i-2} \sum_{n_2 \neq j = n_1}^{i-2} \cdots \sum_{n_{T-1} \neq j = n_{T-2}}^{i-2} q_{n_1} q_{n_2} \ldots q_{n_T} +$$

$$+ 2q_j \Bigg(q_{i-1}^{T-2} + q_{i-1}^{T-2} \sum_{n_1 \neq j = 0}^{i-2} q_{n_1} + \ldots + q_{i-1}^{0} \sum_{n_1 \neq j = 0}^{i-2} \sum_{n_2 \neq j = n_1}^{i-2} \cdots$$

$$\cdots \sum_{n_{T-2} \neq j = n_{T-3}}^{i-2} q_{n_1} q_{n_2} \ldots q_{n_{T-2}} \Bigg) + \ldots + (T-1)\, q_j^{T-2} \times$$

$$\times \Bigg(q_{i-1} + q_{i-1}^{0} \sum_{n_1 \neq j = 0}^{i-2} q_{n_1} \Bigg) + T\, q_j^{T-1} \Bigg] - \frac{F}{p_j}, \quad \text{if } j \leqslant i - 2,$$

where $A = (1 - \alpha_0)(1 - \alpha_1) \ldots (1 - \alpha_{i-1}) p_0 p_1 \ldots p_{i-1}$

$$\frac{\partial P^t_{i,x}}{\partial q_{i-1}} = A \left[T q_{i-1}^{T-1} + (T-1) q^{T-2} \sum_{n_1=0}^{i-2} q_{n_1} + \ldots + \right.$$

$$\left. + \sum_{n_1=0}^{i-2} \sum_{n_2=n_1}^{i-2} \ldots \sum_{n_{T-1}=n_{T-2}}^{i-2} q_{n_1} q_{n_2} \ldots q_{n_{T-1}} \right] - F/p_{i-1},$$

$$\frac{\partial P^t_{i,x}}{\partial \alpha_j} = -F/(1 - \alpha_j) \quad \text{for} \quad j \leqslant i - 1;$$

when $x > m$

$$\frac{\partial P^t_{i,x}}{\partial q_j} = \frac{\partial F}{\partial q_j}\left((\alpha_i + (1 - \alpha_i) q_i^{x-h+1}\right), \quad \text{if} \quad j \leqslant i - 1$$

where $\dfrac{\partial F}{\partial q_j} = \dfrac{\partial P^t_{i,x}}{\partial q_i}$ as obtained when $x \leqslant m$,

$$\frac{\partial P^t_{i,x}}{\partial q_i} = F\left((1 - \alpha_i)(x - h + 1) q_i^{x-h}\right),$$

$$\frac{\partial P^t_{i,x}}{\partial \alpha_j} = -\frac{F}{(1 - \alpha_j)}\left(\alpha_i + (1 - \alpha_i) q_i^{x-h+1}\right), \quad \text{if} \quad j \leqslant i - 1$$

$$\frac{\partial P_{i,x}}{\partial \alpha_i} = F\left(1 - q_i^{x-h+1}\right).$$

Numerical values of these derivatives and $P^t_{i,x}$'s can be obtained with the help of the pilot estimates. Minimizing equations (Equation 7 of the text) can then be utilized to calculate the final estimates. These equations are

$$\sum_{s=1}^{r} (\theta_s - \bar{\theta}_s) \sum_i \sum_x \frac{b_{ixK}\, b_{ixs}}{N_{ix}} = \frac{\sum_i \sum_x b_{ixK}}{N} - \sum_i \sum_x \frac{b_{ixK}\, P^t_{i,x}}{N_{ix}}$$

for $K = 1, 2, \ldots, r,$

$\theta_1, \theta_2 \ldots \theta_r$ are the r parameters (q_i and α_j for $j = 0, 1, \ldots, Z$) of the model and $\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_r$ are the respective pilot estimates.