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Reconciliation of the Integral Equation and the Projection Techniques in the Age-Parity Specific Stable Population

Introduction

IT has long been recognized that current demographic conditions are not well reflected by current crude rates of birth, death, and growth as they are influenced by the past history of fertility and mortality. In order to measure present rates free from the effects of the past, Lotka (1922) developed the stable population theory, showing that if a population maintained unchanging age-specific fertility and mortality rates for an indefinite period, it would eventually have constant rates—birth, death and growth rates, and a constant age structure. These constant rates, called intrinsic rates, are independent of the initial population and, therefore, reflect the current situation more faithfully than the corresponding current crude rates.

A large body of literature has developed expressing dissatisfaction with the use of age-specific fertility and mortality rates as the basic parameters in the Lotka model. Wicksell (1931) used age-marital status specific fertility rates to derive a nuptially controlled net reproduction rate. Arguing that a woman's past fertility behavior is an indicator of her future behavior, Quensel (1939) proposed extension of the specification of fertility by age and parity. Whelpton (1946) used the age-parity model to obtain adjusted rates for the United States. Hyrenius (1948) extended Wicksell's analysis of the nuptiality table to take account of the dissolution of marriages. Bourgeois-Pichat (1948) used the fertility rates specific for age, marital status, and age at marriage to find three parameters that effectively summarize them. Murphy (1966) devised ingenious projection methods for computing intrinsic rates from the age-parity and the age-parity-nuptiality specific stable population models. Recently, Oechsli (1975) has studied the same models using the multiple decrement life table technique.

The literature, however, is still scanty in providing a mathematical formulation to the generalization of Lotka's model from the points of view of the so-called integral equation approach and the projection approach. This paper, therefore, attempts to bridge this gap by systematically developing the age-parity specific model.

1. Review of Lotka's Model

Integral Equation Approach

Let *p(a)* be the probability that a newborn girl child will survive to age *a,* and *m(a)da* the probability that a woman aged *a* will have a girl child born in the age interval *(a, a+da).* Assuming constant fertility and mortality conditions, the number of female births *B(ri)* at time *t* can be expressed as

$$B(t) = \sum_{i=1}^{\infty} K_i e^{r_i t}, \tag{2.1}$$

where *K's* are constants independent of *t* but dependent on the parameters of the model and on the initial population at *t* =0, r_t is the only real root, and r_z, r_s, . . . are the infinite number of complex roots of the integral equation

$$\Psi(r) = \int_0^{\infty} e^{-ra} p(a) m(a) da = 1. \tag{2.2}$$

As $t \to \infty$, $B(t) \to K_1 e^{r_1 t}$; therefore, the only real root r_1 of (2.2) is the intrinsic growth rate. Substituting r_1 for r, we can obtain the intrinsic birth rate b

and the stable age distribution $c\,(a)$ from the expressions

$$b = 1 \Big/ \int_0^\infty e^{-ra} p(a)da; \quad c(a) = be^{-ra}p(a). \tag{2.3}$$

The net reproduction rate N and the gross reproduction rate G for females are given by

$$N = \int_0^\infty p(a)m(a)da; \qquad G = \int_0^\infty m(a)da. \tag{2.4}$$

To compute the intrinsic rates from five-year age groups, assuming the childbearing years to be 15-50, $\Psi\,(r)$ in (2.2) can be expressed approximately as

$$\Psi(r) = \sum_{x=3}^{9} e^{-(5x+2.5)r} m(5x + 2.5)\,{}_5L_{5x}/l_0 = 1, \tag{2.5}$$

where $m\,(x)$ is the female age-specific fertility rate for the age group $(x - 2.5, x + 2.5)$, and ${}_5L_x$ and l_0 are the female life table quantities with their usual meanings. Starting with an arbitrary value r_0 for r, we can find the real root of (2.5) iteratively by the Newton-Raphson method :

$$r_{i+1} = r_i + [1 - \Psi(r_i)]/\Psi'(r_i). \tag{2.6}$$

If we assume that 100 years is the upper limit of the human life span, the birth rate b and the proportion $c\,(x, x + 5)$ of the female population in the age group $(x, x + 5)$ can be obtained from (2.3) :

$$b = l_0 \Big/ \sum_{x=0}^{19} e^{-(5x+2.6)r}\,{}_5L_{5x}, \tag{2.7}$$

and

$$c\,(5x, 5x + 5) = be^{-(5x+2.5)r}\,{}_5L_{5x}/l_0, \quad x=0, 1, \ldots, 19. \tag{2.8}$$

The net and gross reproduction rates in (2.4) can be calculated from the discrete data by the formulas

$$N = \sum_{x=3}^{9} m\,(5x + 2.5)\;{}_5L_{5x}\Big/ l_0; \quad G = 5 \sum_{x=3}^{9} m\,(5x + 2.5). \qquad (2.9)$$

Projection Approach

Assuming that $P\,(5x,\, 5x + 5,\, t)$ is the female population in the age group $(5x,\, 5x + 5)$ at time t, $x = 0, 1, \ldots, 19$, we can obtain the populations in the age groups from 5-100 at time $t + 5$ from

$$P\,(5x,\, 5x + 5,\, t + 5) = P\,(5x - 5,\, 5x,\, t)\;{}_5L_{5x}/{}_5L_{5x-5},$$
$$x = 1, 2, \ldots, 19. \qquad (2.10)$$

To obtain the number of female births $B\,(t,\, t + 5)$ in the five-year period $(t,\, t + 5)$, we assume that the average number of women in the age group $(5x,\, 5x + 5)$ in this period is given by the geometric mean of the number of women in that age group at the beginning and end of the period. We, therefore, have

$$B(t,\, t + 5) = 5 \sum_{x=3}^{9} m(5x + 2.5)\;\sqrt{P(5x,\, 5x + 5,\, t)\,P(5x,\, 5x + 5,\, t + 5)}. \qquad (2.11)$$

If we break $B\,(t,\, t + 5)$ down into $B\,(t + h - 1,\, t + h)$'s, $h = 1, 2, \ldots, 5$, we will get the female births in one-year periods. Denoting the total female population in the childbearing years 15-50 at time t, by $P^*\,(t)$, we have, approximately,

$$B(t + h - 1,\, t + h) = B(t,\, t + 5)\,\frac{[P^*(t)]^{(1.1-.2h)}\,[P^*(t + 5)]^{(.2h-.1)}}{\sum_{h=1}^{5}\,[P^*(t)]^{(1.1-.2h)}\,[P^*(t + 5)]^{(.2h-.1)}},$$
$$h = 1, 2, \ldots, 5. \qquad (2.12)$$

The population $P\,(0,\, 5,\, t + 5)$ can be expressed as the sum of the populations in the age groups $(0, 1)$ and $(1, 5)$ at time $t + 5$ by

$$P(0,\, 5,\, t + 5) = B(t + 4,\, t + 5)\;{}_1L_0/l_0 + [B(t,\, t + 5) - B \times$$
$$\times\,(t + 4,\, t + 5)]\;{}_4L_1/(4l_0). \qquad (2.13)$$

If we denote the total female population at time t by $P(t)$, the average annual growth rate $r(t, t+5)$ and the average annual birth rate $b(t, t+5)$ in the period $(t, t+5)$ can be calculated from

$$r(t, t+5) = [\log P(t+5) - \log P(t)]/5,$$

$$b(t, t+5) = .2B(t, t+5)/\sqrt{P(t)P(t+5)}. \qquad (2.14)$$

The projections are continued until r and b in (2.14) become stable up to a specified number of decimal places. When this situation is reached, these values should be sufficiently close to those which were obtained by the integral equation approach in (2.6) and (2.7).

Age-Parity Model

Integral Equation Approach

As before, we denote by $p(a)$ the probability that a newborn girl child will survive to age a, and this probability is assumed to be independent of her parity history up to that age. The probability that a woman of age a and parity n (including her sons) will have a child born in the age interval $(a, a + da)$ is denoted by $m(a, n)\, da$. Let us also assume that the probability of a child being born female is a constant independent of mother's age, and let us denote this by θ_F. Thus, when $t \to \infty$, the number of female births $B(t)$ at time t can be expressed as

$$B(t) = \int_0^t \Big[\sum_{n=0}^{z-1} B(t-a)\, p(a)\, R(a, n)\, m(a, n)\theta_F \Big] da, \qquad (3.1)$$

where z is the largest possible parity and $R(a, n)$ is the probability (at birth) that a woman will be of parity n at age a, given that she will survive to age a. Assuming the childbearing years to be 15-50, we have

$$R(a, 0) = e^{-\int_{15}^{a} m(t, 0)dt},$$

$$R(a, n) = \int_{15}^{a} R(u, n-1)\, m(u, n-1) e^{-\int_{u}^{a} m(t, n)dt}\, du, \quad n = 1, 2, \ldots, z-1,$$

and

$$R(a, z) = \int^{a} R(u, z-1)\, m(u, z-1) du. \qquad (3.2)$$

Substituting $B(t) = Ke^{rt}$ in (3.1), it follows that the intrinsic growth rate is the only real root of the integral equation

$$\Psi^*(r) = \int_0^\infty e^{-ra} p(a)\, m^*(a) da = 1, \tag{3.3}$$

where

$$m^*(a) = \theta_F \sum_{n=0}^{z-1} R(a, n)\, m(a, n). \tag{3.4}$$

$m^*(a)$ is the *intrinsic* female fertility rate at age a, whereas $m(a)$ is the corresponding *observed* rate in Lotka's model. Comparing (2.2) and (3.2), we notice that the computation of the intrinsic rates r, b, $c(a)$, N, and G, in the age-parity model is analogous to their computation in the age model, except that in the former the rates $m^*(a)$ are used instead of $m(a)$. Thus, the main problem in the age-parity model is the calculation of the rates $m^*(a)$ or, in other words, the probabilities $R(a, n)$.

It is obvious from (2.5) and (3.4) that to compute rates from five-year age-group data we need the values of $R(a, n)$ only for $a = 17.5, 22.5, \ldots, 47.5$. Since it is not easy to solve directly for $R(a, n)$ from (3.2), we can do it successively from the following three sets of relationships :

I. $R(a, n) = \sum_{s=0}^{n} R(a-5, s)\, Q(a, n \mid a-5, s)$, $a = 22.5, \ldots, 47.5$, (3.5)

$$R(17.5, n) = Q(17.5, n \mid 12.5, 0),$$

II. $Q(a, n \mid a-5, s) = \sum_{w=s}^{n} Q(a-2.5, w \mid a-5, s)\, Q(a, n \mid a-2.5, w)$,

$a = 22.5, \ldots, 47.5$, $Q(17.5, n \mid 12.5, 0) = Q(17.5, n \mid 15.0, 0)$. (3.6)

III. $Q(a, n \mid a-2.5, w) = Q(a+2.5, n \mid a, w)$, $a = 17.5, \ldots, 47.5$, (3.7)

where $Q(u, s \mid a, n)$ is the probability that a woman of age a and parity n will be of parity s at age u, given that she will survive to age u. Thus, the whole problem boils down to finding all possible values of $Q(a + 2.5, s \mid a, n)$ for $a = 17.5, 22.5, \ldots, 47.5$.

If ages a and u both lie in the same five-year age-group $(5x, 5x+5)$, $x = 3, 4, \ldots, 9$, then we have the integral equations

$$Q(u, n \mid a, n) = e^{-(u-a)m(5x+2.5,n)}, \quad n < z,$$

$$Q(u, s \mid a, n) = \int_a^u Q(v, s-1 \mid a, n)\, m(v, s-1) e^{-(u-v)m(5x+2.5,s)} dv, \quad n < s < z,$$

$$Q(u, z \mid a, n) = \int_a^u Q(v, z-1 \mid a, n)\, m(v, z-1)\, dv, \quad n < z, \qquad (3.8)$$

and

$$Q(u, z \mid a, z) = 1.$$

Solving (3.8), we obtain the required values :

$$Q(a+2.5, n \mid a, n) = e^{-2.5m(a,n)}, \quad n < z,$$

$$Q(a+2.5, s \mid a, n) = \left[\prod_{i=n}^{s-1} m(a, i)\right] \sum_{i=n}^{s} \left[e^{-2.5m(a,i)} \Big/ \prod_{\substack{j=n \\ j \neq i}}^{s} \{m(a, j) - m(a, i)\} \right], \quad n < s < z,$$

$$Q(a+2.5, z \mid a, n) =$$

$$= \sum_{i=n}^{z-1} \left[1 - e^{-2.5m(a,i)}\right] \left[\prod_{\substack{j=n \\ j \neq i}}^{z-1} m(a, j) \Big/ \prod_{\substack{j=n \\ j \neq i}}^{z-1} \{m(a, j) - m(a, i)\} \right], \quad n < z,$$

and

$$Q(a+2.5, z \mid a, z) = 1, \quad \text{for } a = 17.5, 22.5, \ldots, 47.5. \qquad (3.9)$$

Projection Approach

Using the same notations as in the projection approach for the Lotka model, we obtain the populations at time $t + 5$ in the age groups from 5-100 by (2.10). The crucial step, therefore, is to find the number of female births $B(t, t+5)$ in the period $(t, t+5)$.

Let us denote by $F(5x, 5x+5, n, t)$ the female population of parity n in the age group $(5x, 5x+5)$ at time t, $x = 3, 4, \ldots, 9$, $n = 0, 1, \ldots, z$. It follows that

$$F(5x, 5x+5, n, t+5) =$$

$$= \left[\sum_{s=0}^{n} F(5x-5, 5x, s, t)\, Q(5x+2.5, n \mid 5x-2.5, s)\right] {}_5L_{5x} \Big/ {}_5L_{5x-5},$$

$$x = 4, 5, \ldots, 9, \qquad (3.10)$$

and

$$F(15, 20, n, t+5) = P(15, 20, t+5)\, Q(17.5, n \mid 12.5, 0).$$

Assuming again, that the average number of women of parity n in the age group $(5x, 5x+5)$ in the period $(t, t+5)$ is given by the geometric mean of the number of such women at the beginning and end of the period, we have the expression

$$B(t, t+5) = 5 \sum_{x=3}^{9} \sum_{n=0}^{z-1} m(5x+2.5, n) \sqrt{F(5x, 5x+5, n, t) F(5x, 5x+5, n, t+5)}. \qquad (3.11)$$

Once $B(t, t+5)$ is obtained, $P(0, 5, t+5)$ and the trajectories of r and b can be found by expressions analogous to (2.12)-(2.14). As in the case of age model, we expect that the formulas for integral equation and projection approaches in the age-parity model will also give virtually identical intrinsic rates.

It should be noted that the formulas (3.5) to (3.11) are appropriate only when data are available in five-year age groups. The same expressions, however, can be used for any other age grouping of width W by substituting W and $W/2$ for 5 and 2.5 respectively.

We have used our expressions to obtain the results from the age and age-parity models for the U.S. female population, 1970. The observed growth and birth rates per 1,000 women are 9.31 and 17.36 in 1970. Both integral equation and projection approaches in the age model give the intrinsic growth and birth rates (per 1,000 women), and the gross and net reproduction rates (per woman) as 5.74, 16.63, 1.20, and 1.16 respectively. The same rates in the age-parity model turn out to be 6.01, 16.80, 1.21, and 1.17 respectively by both approaches.

Table 1 gives the trajectories of growth rate, birth rate, and the female populations in the period 1970-2200, as obtained by the two models by projecting the U.S. female population with constant fertility and mortality rates as of

TABLE I—PROJECTION OF U.S. FEMALE POPULATION, 1970-2200, WITH FERTILITY AND MORTALITY PARTS OF 1970

Period (t, $t+5$)	*Age Model*			*Age-Parity Model*		
	Average Annual Rates per 1,000 Women		*Female Population at Time t + 5 (millions)*	*Average Annual Rates per 1,000 Women*		*Female Population at Time t + 5 (millions)*
	Growth	*Birth*		*Growth*	*Birth*	
1970	9.31	17.36	105	9.31	17.36	105
1970-75	9.97	18.04	110	10.05	18.11	110
1975-80	10.16	19.05	116	10.43	19.32	116
1980-85	9.83	18.98	122	10.01	19.14	122
1985-90	8.61	17.97	127	8.70	18.03	127
1990-95	7.61	17.10	132	7.71	17.17	132
1995-00	7.38	17.p3	137	7.55	17.16	137
2000-05	7.60	17.30	142	7.84	17.50	*143*
2005-10	7.65	17.34	148	7.89	17.53	149
2010-15	7.30	17.06	153	7.50	17.20	154
2015-20	6.94	16.74	158	7.14	16.86	160
2020-25	6.77	16.61	164	6.99	16.76	166
2025-30	6.50	16.63	169	6.76	16.81	171
2030-35	6.11	16.67	175	6.39	16.85	177
2035-40	5.61	16.65	180	5.88	16.81	182
2040-45	5.51	16.60	185	5.78	16.75	187
2045-50	5.45	16.60	190	5.72	16.75	193
2070-75	5.61	16.63	219	5.87	16.78	224
2095-00	5.70	16.62	253	5.96	16.79	261
2120-25	5.73	16.63	292	5.99	16.79	303
2145-50	5.74	16.63	337	6.00	16.80	352
2170-75	5.74	16.63	389	6.01	16.80	409
2195-00	5.74	16.63	449	6.01	16.80	475
∞	5.74	16.63	∞	6.01	16.80	∞

1970. As expected, the growth and birth rates eventually converge to theft respective intrinsic rates in both age and age-parity models.

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