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Some Characteristics of a Two-Sex Stable Population Model

1. Two-Sex Models

ONE of the major drawbacks of the theory of stable population stems from the fact that the assumption of the continuity of the given sets of mortality and fertility rates usually results in different intrinsic rates of growth for the two sexes (Karmel, 1947, 1948). Although mortality rates of the two sexes can properly be regarded as virtually independent of one another, the phenomenon of reproduction, being truly dependent upon the cooperation between the males and females, must be expressed in a manner that recognizes this relationship in some form or the other. This was attempted earlier by Knibbs (1911), who proposed a symmetric but nonlinear relationship to describe number of marriages between men and women of different age categories. Recent attempts (Feeney, 1972 ; Keyfitz, 1968, 1971 ; MacFarland, 1970, 1972 ; Parlett, 1972 and Pollard, 1971), produced different marriage functions with appropriate consistency conditions but most of these suffer from difficulties in empirical determination of function parameters.

Das Gupta (1972) proposed a function by the name of effective population which was constructed without the explicit acknowledgement of marriage as a social phenomenon. The fertility rates specific for the two sexes and their age compositions were then defined. Das Gupta then demonstrated that the constancy of these two-sex specific rates together with the constancy of age specific mortality rates of the two sexes over time, resulted in identical intrinsic rates of growth for both males and females. However, it was observed that this rate did not always lie within the interval defined by the similar rates of the two one-sex models, a requirement that was suggested by Coale (1972), as desirable.

Recently, Mitra (1975) has advanced the idea of constructing a fertility model as an explicit function of the age of any one of the two sexes. For such a model, he has shown that the intrinsic rate of growth, common for the two sexes, also meets the boundary condition suggested by Coale. The definitions of the functions and some of the findings of that investigation are summarized below for the sake of continuity. Further development of the model is carried out in subsequent sections.

2. Birth Function

Let $B_M(a, t)$ be the number of male births to the male population aged a at time t, namely $M(a, t)$. It is assumed that $B_M(a, t)$ is proportional to (i) $M(a, t)$, and also to (ii) the proportion of the female to the total population (male and female) in reproductive ages denoted by $u(t)$. Combining the two, $B_M(a, t)$ can be expressed as

$$B_M(a, t) = K_M(a, t)\, M(a, t)\, u(t), \tag{1}$$

where $K_M(a, t)$ accounts for the contribution of other factors towards the reproductive behavior. The male age-specific birth rate can then be expressed as

$$m_M(a, t) = \frac{B_M(a, t)}{M(a, t)} = K_M(a, t)\, u(t), \tag{2}$$

and the same for the female counterpart as

$$m_F(a', t) = \frac{B_F(a', t)}{F(a', t)} = K_F(a', t)\, v(t), \tag{3}$$

$$u(t) + v(t) = 1. \tag{4}$$

where $v(t)$ is the proportion of the male to the total reproductive population. It may be pointed out that, from a mathematical point of view, $u(t)$ and $v(t)$ need not be explicitly determined. One of the logical conditions that follows from their definitions is

An initial value of $u(t)$ and $v(t)$ may, however, be obtained by defining the sum' of all females in the age interval, say 15-49 and all males in the 15-54 age interval as the total reproductive population. It should also be noted that given $K_M(a, t)$ and $K_F(a', t)$, the condition that $u(t)$ and v(t) must satisfy, in addition to (4), is

where S, the sex-ratio at birth, is a constant, independent of time. The consequences on the age-sex composition of the assumptions of the continuity of a set of KM and KF values (derived for say, an initial time period from given birth data and some u), together with the continuity of the life tables of the two sexes were examined next. Denoting by $PM(A)$, the probability of a male surviving from birth to age a and by $p_F(a')$, the corresponding probability of the female, it was shown that the intrinsic rate of growth r for the two-sex model can be obtained as a solution of the equation (see Appendix)

$$\frac{1}{\int_0^\infty e^{-ra} p_M(a) K_M(a)\,da} + \frac{1}{\int_0^\infty e^{-ra'} p_F(a') K_F(a')\,da'} = 1. \tag{6}$$

Further, this value of r was found to be included in the interval $(R_M r_F)$, where r_M and r_F are the male and female intrinsic rates of growth obtained from the respective one-sex stable population models.

3. Effect of the Composition of the Reproductive Population

Since a measure of the sex composition of the reproductive population is provided by $u(t)$, it is natural to expect that $u(t)$ will have some influence on the estimate of r. This becomes clear when for, say, t — 0, (6) is written as

$$\frac{u(0)}{\int_0^\infty e^{-ra} p_M(a) m_M(a, 0)\,da} + \frac{v(0)}{\int_0^\infty e^{-ra'} p_F(a') m_F(a', 0)\,da'} = 1, \tag{7}$$

utilizing the relationship between m, K and u functions given in (2) and (3). The denominator of $u(0)$ in (7) can be expressed as

$$\int_0^\infty e^{-(r-r_M)a} e^{-r_M a} p_M(a) m_M(a, 0) da \tag{8}$$

and then as

$$e^{-(r-r_M)\bar{a}} \int_0^\infty e^{-(r-r_M)(a-\bar{a})} e^{-r_M a} p_M(a) m_M(a, 0)\, da, \tag{9}$$

where $\bar{a}$ is the average age of childbearing in the male stable population for which the intrinsic rate of growth r_M is the real root of the integral equation

$$\int_0^\infty e^{-r_M a} p_M(a) m_M(a, 0) da = 1. \tag{10}$$

Expansion of the exponential $e^{-(r-r_M)(a-\bar{a})}$ in the integral factor of (9) and term-by-term integration produces

$$1 + \frac{(r - r_M)^2}{2!} V_M(a) - \ldots \tag{11}$$

in which $V_M(a)$ is the variance of the age distribution of childbearing shown in (10). Since $r - r_M$ is small (rarely greater than .005) and $\sqrt{V_M(a)}$ rarely greater than 10 years, contributions of terms of the $0(r^2)$ in (11) can be ignored. Therefore,

$$e^{-(r-r_M)\bar{a}} \tag{12}$$

is a sufficiently close approximation of (9).

Simplifying the denominator of $v(0)$ in (7) in the same manner and combining, (7) can be transformed into

$$u(0)e^{(r-r_M)\bar{a}} + v(0)e^{(r-r_F)\bar{a}'} = 1, \tag{13}$$

where $\bar{a}'$ is the average age of childbearing in the female stable population. Since the average ages of childbearing are generally less than 30 years, terms of the $0(r^2)$ can also be ignored from the expansion of the exponentials in (13), so that an approximate solution of r can be obtained from

$$u(0)[1 + (r - r_M)\bar{a}\,] + v(0)[1 + (r - r_F)\bar{a}'] = 1. \tag{14}$$

Because of (4), the solution of r can be expressed as

$$r = \frac{u(0)\bar{a}r_M + v(0)\bar{a}'r_F}{u(0)\bar{a} + v(0)a'} \tag{15}$$

and also as

$$r = r_M + \frac{r_F - r_M}{1 + \dfrac{u(0)}{v(0)}\dfrac{\bar{a}}{\bar{a}'}}. \tag{16}$$

Compared with the solution presented earlier by Mitra (*ibid.*), the present solution seems to be more accurate and also much simpler in form. It may also be noted that r lies in the interval (r_F, r_M), and that its estimate depends not only on $u(0)$, $v(0)$, r_M and r_F, which are expected, but also on a and $\bar{a}'$. The dependence on the latter two parameters is not unwarranted in view of their close relationship with the lengths of generation which, in turn, are related to the intrinsic rate of growth through the respective net reproduction rates.

4. Choice of $u(0)$ and $v(0)$

No single rule or a set of rules can be prescribed for choosing $u(0)$ at the initial time period. However, there is at least one case that deserves special attention. From the point of view of stability, it is imperative that $u(t)$ and $v(t)$ also approach stable values which, in turn, will depend on their initial values, namely $u(0)$ and $v(0)$.

Since $u(t) + v(t) = 1$, the special case of interest, naturally, is the situation that results in equal stable values of $u(t)$ and $v(t)$ as $t \to \infty$, i.e., when $u = v = 1/2$. From purely logical considerations, such a requirement seems to be justified, especially in monogamous societies, in which the relationship between the two sexes can be regarded as symmetrical, at least when stability has been achieved. At the initial time period, when the age composition differs from its intrinsic pattern, one may describe the situation as lacking in stability, as an explanation of the fluctuating values of the required parameters, including *u(t)* and v(t)

Accordingly, the suggestion that the initial values of u(0) and v(0) be chosen in a manner so that under conditions of stability *u(t)* and *v(t)* should approach equality, deserves special consideration. The procedures for obtaining these values, together with the corresponding solution of *r* have been outlined next.

5. Derivation of the Stable Parameters

For the derivation of the stable parameters, one may begin by noting that in the resulting stable population, the relation

$$\int_0^\infty e^{-ra} p_M(a) m_M(a)\,da = \int_0^\infty e^{-ra'} p_F(a') m_F(a')\,da' = 1 \qquad (17)$$

must hold, in which r is the common intrinsic rate of growth for the two sexes and $m_M(a)$, $m_F(a')$ are the limiting values of the respective age-specific fertility rates, i.e., as $t \to \infty$, $m_M(a, t) \to m_M(a)$ and $m_F(a', t) \to m_F(a')$.

Similarly, denoting the limiting stable values of $u(t)$ and $v(t)$ by u and v, (2) and (3) can be rewritten as

$$K_M(a) = \frac{m_M(a, 0)}{u(0)} = \frac{m_M(a)}{u}, \qquad (18)$$

$$K_F(a') = \frac{m_F(a', 0)}{v(0)} = \frac{m_F(a')}{v}, \qquad (19)$$

when the *K* functions are assumed to be independent of time. Accordingly, (17) can be written as

$$\frac{u}{u(0)} \int_0^\infty e^{-ra} p_M(a) m_M(a, 0)\,da = \frac{v}{v(0)} \int_0^\infty e^{-ra'} p_F(a) m_F(a', 0)\,da' = 1. \quad (20)$$

Taking advantage of the simplifications achieved for (7) and presented in (14), the following two identities are generated from (20), namely

$$u = u(0)[1 + (r - r_M)\bar{a}] \tag{21}$$

and

$$v = v(0)[1 + (r - r_F)\bar{a}']. \tag{22}$$

These expressions of *u* and v for given values of u(0), v(0) and the parameters derivable from the initial distributions, are quite general from which special cases can be easily deduced. As an example, for *u* = v = 1/2,

$$u(0) = \tfrac{1}{2}[1 - (r - r_M)\bar{a}] \tag{23}$$

and

$$v(0) = \tfrac{1}{2}[1 - (r - r_F)\bar{a}'], \tag{24}$$

where terms of the $0(r^2)$ are ignored. Adding (23) and (24) and simplifying,

$$(r - r_M)\bar{a} + (r - r_F)\bar{a}' = 0, \tag{25}$$

because of (4), from which

$$r = r_M + \frac{r_F - r_M}{1 + \bar{a}/\bar{a}'} \tag{26}$$

is obtained. Substitution of *r* from (26) in (23) and (24) will produce w(0) and v(0). An interesting comparison can be made between this particular solution of *r* with the general solution provided in (16). Note also that the substitution of u(0) = v(0) in that general solution produces a value of *r,* that is, the same as (26). However, *u* and v are different as they should be, and are given by

$$u = \tfrac{1}{2}[1 + (r - r_M)\bar{a}] \tag{27}$$

and

$$v = \tfrac{1}{2}[1 + (r - r_F)\bar{a}'], \tag{28}$$

respectively, as the values $u(0) = v(0) = \frac{1}{2}$ are substituted in (21) and (22). It must be made clear that this equality of the two solutions of r is approximate, since in both cases $u/u(0)$ assumes identical value because terms of the $0(r^2)$ are ignored. This distinction, though subtle and somewhat unimportant in terms of practical usage, must be made from mathematical points of view.

The discussion can now be closed by pointing out the relationships between the limiting values of the net reproduction rates R_{OM} and R_{OF} with those at the initial time period, namely, $R_{OM}(0)$ and $R_{OF}(0)$. By definition,

$$R_{OM} = \int_0^\infty p_M(a) m_M(a) da = \frac{u}{u(0)} R_{OM}(0) \tag{29}$$

because of (18). Similarly,

$$R_{OF} = \int_0^\infty p_F(a') m_F(a') da' = \frac{v}{v(0)} R_{OF}(0). \tag{30}$$

The lengths of generation for the males and females L_M and L_F can then be obtained for the stable population as

$$L_M = \frac{\log_e R_{OM}}{r} \tag{31}$$

and

$$L_F = \frac{\log_e R_{OF}}{r}. \tag{32}$$

6. Discussion

The above stable population model depends; in part, on the mortality and fertility rates specific for both age and sex. This is expected. In addition, and this is somewhat crucial, the model also depends upon the choice of a specific set of $u(t)$ and $v(t)$ where these provide a measure of the sex composition of the population. The values of $u(t)$ and $v(t)$ assumed for the initial time period, say, $t = 0$, ate found to be ie\ated to the« ultimate 01 stable values u and v. It has been shown that the ratios $u/u(0)$ and $v/v(0)$ are significant determinants of the parameters of this stable population model.

From logical considerations derived from a symmetrical relationship between the two sexes, $u(t)$ and $v(t)$ can be assigned values in close proximity of .5. In that narrow range, the parameters of the model are virtually invariant because of virtual constancies of the ratios $u/u(0)$ and $v/v(0)$, and more so, when the one-sex intrinsic rates of growth are not too dissimilar, as is usually the case. In reality, the age-sex-specific birth rates at the initial time period turn out to be quite similar and, in fact, proportional to the corresponding intrinsic values, where the constants of proportionality are, for both sexes, quite close to unity. In summary, then, the principal thesis of this paper has been to demonstrate that the existing sets of age-sex specific fertility rates in a population can be easily converted to those that are internally consistent with one another from the perspective of a stable population.

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APPENDIX

The number of male births in year t, namely, $B_M(t)$ can be expressed as

$$B_M(t) = \int_0^\infty B_M(t-a)p_M(a)m_M(a,t)\,da. \tag{33}$$

Since, $m_M(a,t) = K_M(a)u(t)$

$$\frac{B_M(t)}{\int_0^\infty B_M(t-a)p_M(a)K_M(a)da} = u(t). \tag{34}$$

Writing a similar expression for the females and adding the two

$$\frac{B_M(t)}{\int_0^\infty B_M(t-a)p_M(a)K_M(a)da} + \frac{B_F(t)}{\int_0^\infty B_F(t-a')p_F(a')K_F(a')da'} = 1, \tag{35}$$

because $u(t) + v(t) = 1$. Since $B_M(t) = SB_F(t)$, where S, the sex-ratio at birth, is independent of time, (35) can be written as

$$B_F(t)\left[\frac{1}{\int_0^\infty B_F(t-a)p_M(a)K_M(a)da} + \frac{1}{\int_0^\infty B_F(t-a')p_F(a')K_F(a')da'}\right] = 1. \tag{36}$$

Following the customary procedure of using a trial solution

$$B_F(t-a) = B_F(t)^{e^{-ra}}, \tag{37}$$

(36) is reduced to (6).