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Research Article

Multivariate Force of Mortality

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Abstract

In usual demographic analysis, force of mortality is a function of one variable, that is, of age. In this article bi-variate and multivariate force of mortality functions are introduced for the first time to explain mortality differentials. The pattern of mortality in a population is one of the strong influencing factors in determining the life expectancies at various ages in the population. Considering univariate functions of age only to understand the human mortality data without associating with other variables could lead to incomplete analysis. The reasons behind declining forces of mortality globally could be studied using the proposed functions. Other applications of multivariate forces of mortality could be in actuarial sciences.

Introduction

One variable force of mortality, $\mu(x)$, with respect to age, x of an individual is one of the central topics of study in the actuarial mathematics and there are very useful discussions are available on univariate or one variable force of mortality (see for example, Smith, 1948; Turner et al. 2010; Finkelstein, 2003). It is often termed as instantaneous rate of death at an age x. Suppose, forces of mortality is measured on two variables (x,y), (one being age, other can be some influencing factor on mortality), then if we plot the force of mortality $\mu(x,y)$ on the $xy-$plane, then for an arbitrary point (x_0,y_0), we can write,

$$\mu\left(x_0,y_0\right)=\frac{1}{\rho(\Omega)}\iint_{\Omega}\mu(x,y)d\rho \tag{1}$$

where Ω is a region such that $\left(x_0,y_0\right)\in\Omega$ and $\rho(\Omega)$ is area of the region. Since μ is continuous in the univariate case, if we assume the same holds for $\mu\left(x,y\right)$ in Ω, it has an upper bound say, μ_1 and a lower bound say, μ_0 in the region such that

$$\mu_0\leq\frac{1}{\rho(\Omega)}\iint_{\Omega}\mu(x,y)d\rho\leq\mu_1.$$

Multivariate force of mortality functions can help in better understanding of longevity and causes of decline in mortality rates. There are studies which consider mortality decline or longevity projections of humans with respect to age only (for example, see Turner et al. 2010; Finkelstein, 2003). Such studies can be handled using univariate analysis of standard force of mortality functions. For capturing simultaneous dynamics in mortality rates or causes of increase in longevity we have introduced multivariate forces of mortality functions. Analytical properties, such as the rate of change in forces of mortalities of bi-variate force of mortality functions are found in the next sections. In this section, we will begin with two examples and we illustrate their numerical simulations.

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Example 1. *Suppose* $l(x,y)=1-x^a y^{b/\sqrt{K}}/K$ *for* $a,b,K\in\mathbb{Z}^+$. *See the definitions of* $\mu_x(x,y)$ and $\mu_y(x,y)$ in (20) and (21) in the section 3. We have,

$$\mu_x(x,y)=\frac{ax^{a-1}y^{b/\sqrt{K}}}{K-x^a y^{b/\sqrt{K}}} \quad \text{and} \quad \mu_y(x,y)=\frac{bx^a y^{(b/\sqrt{K})-1}}{\sqrt{K}\left(K-x^a y^{b/\sqrt{K}}\right)}$$

Example 2. *Suppose* $l(x,y)=a^{\sqrt{y}}b^{x^3}$ *for* $10\le x\le 80$, $1\le y\le 5$, $a=5$, $b=7$. $\mu_x(x,y)=-3x^2\log(b)$ *and* $\mu_y(x,y)=-\log(a)/2\sqrt{y}$.

Force of mortality functions also associated with continuous life table functions (Smith, 1948). Life table is a mathematical model describing how individuals born at same time (for example, a cohort of new born babies) survive over the years at various ages until the last individual dies. Life table is constructed based on present or past mortality pattern (i.e. mortality rates at each age, (say x) per fixed number of individuals in the same age x per year or for the year $(0,t)$) in the population and assumed that this pattern will remain the same until the individual at last age dies. This table can be used to construct synthetic population at each age x at time t or for time interval $(0,t)$, it do not have mechanism to take care of future changes in the mortality pattern after t. Suppose $l(x)$ denote the number of individuals at age x out of $l(0)$ newly born individuals in a life table with continuous partial derivatives up to order $(k+1)$. Then the number of individuals at ages, $x+\Delta x$ and $x-\Delta x$ are denoted by $l(x+\Delta x)$ and $l(x-\Delta x)$ can be obtained from Taylor series expansion evaluating at x_0 as

$$\begin{aligned} l(x_0+\Delta x)&=l(x_0)+\Delta x l'(x_0)+(\Delta x)^2\frac{l^{(2)}(x_0)}{2!}+(\Delta x)^3\frac{l^{(3)}(x_0)}{3!}+\ldots \\ &+(\Delta x)^k\frac{l^{(k)}(x_0)}{k!}+\int_{x_0}^{x}\frac{(\Delta x)^k l^{(k+1)}(t)}{k!}dt \end{aligned} \tag{2}$$

$$\begin{aligned} l(x_0-\Delta x)&=l(x_0)-\Delta x l'(x_0)+(\Delta x)^2\frac{l^{(2)}(x_0)}{2!}-(\Delta x)^3\frac{l^{(3)}(x_0)}{3!}+\ldots \\ &+(-1)^k(\Delta x)^k\frac{l^{(k)}(x_0)}{k!}+\int_{x_0}^{x}\frac{(\Delta x)^k l^{(k+1)}(t)}{k!}dt \end{aligned} \tag{3}$$

Here x is the fixed age between 0 and ω, the maximum age of life. The number of survivors at age x with some other relevant factor (for example, marital status, education level, climate, food habits, geographic region etc) y evaluated at (x_0,y_0) can be obtained form two variable Taylor expansion. Assuming continuous partial derivatives for $l(x_0+\Delta x,y_0+\Delta y)$ up to order 3, we can the expansion of two variable survival functions as follows:

$$\begin{aligned} l(x_0+\Delta x,y_0+\Delta y)&=l(x_0,y_0)+\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+\Delta y\left\{\frac{\partial l}{\partial y}(x_0,y_0)\right\}+\frac{(\Delta x)^2}{2!}\left\{\frac{\partial^2 l}{\partial x^2}(x_0,y_0)\right\} \\ &+\frac{(\Delta y)^2}{2!}\left\{\frac{\partial^2 l}{\partial y^2}(x_0,y_0)\right\}+\Delta x\Delta y\left\{\frac{\partial^2 l}{\partial x\partial y}(x_0,y_0)\right\}+\frac{(\Delta x)^3}{3!}\left\{\frac{\partial^3 l}{\partial x^3}(x_0,y_0)\right\} \\ &+\frac{(\Delta y)^3}{3!}\left\{\frac{\partial^3 l}{\partial y^3}(x_0,y_0)\right\}+\frac{(\Delta x)^2\Delta y}{2}\left\{\frac{\partial^3 l}{\partial x^2\partial y}(x_0,y_0)\right\}+\frac{\Delta x(\Delta y)^2}{2}\left\{\frac{\partial^3 l}{\partial x\partial y^2}(x_0,y_0)\right\}+\cdots+\cdots \end{aligned} \tag{4}$$

Figure 1

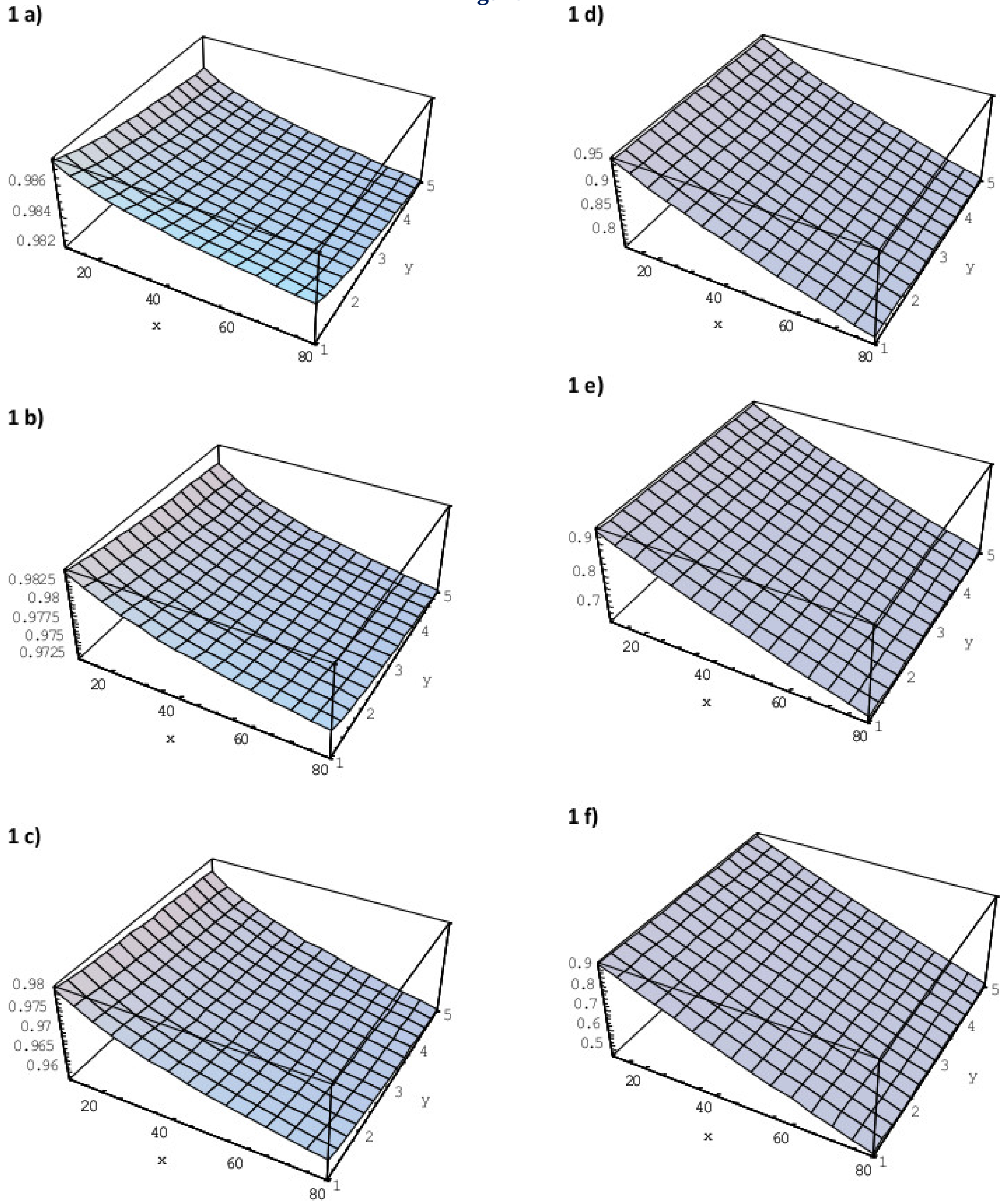

Figure 1. $l(x, y)$ in Example 1. For various combinations of values of **a** and **b** we have drawn Figures **(1 a)** to **(1 f)** by fixing K=100, $10 \leq x \leq 80$ and $1 \leq y \leq 5$.

Following are the combinations of **a** and **b** for each figure: **(1 a)**: a=0.1, b=0.9, **(1 b)**: a=0.2,b=0.8, **(1 c)**: a=0.3, b=0.7, **(1 d)**: a=0.7,b=0.3, **(1 e)**: a=0.8,b=0.2, **(1 f)**: a=0.9,b=0.1.

Figure 2

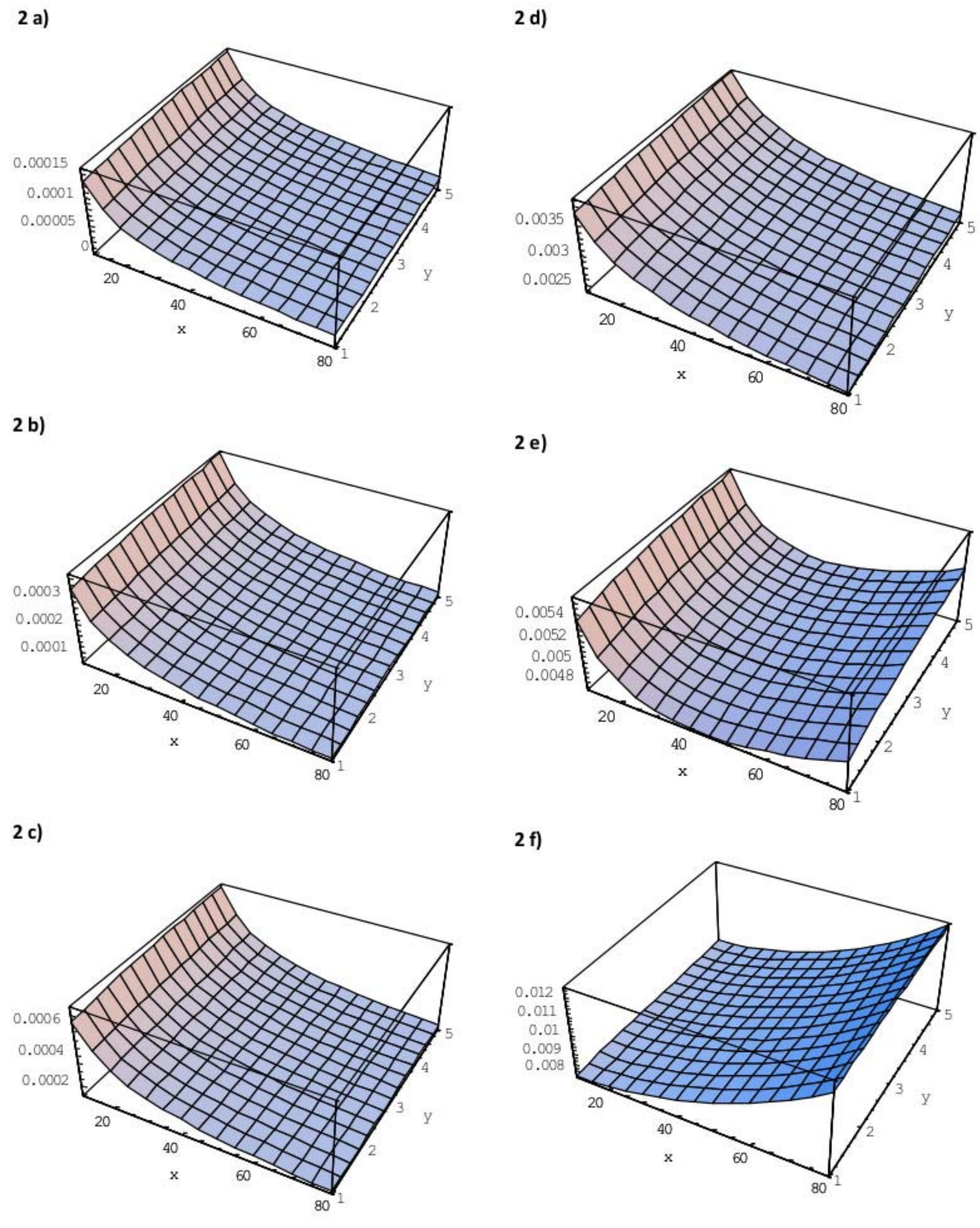

Figure 2. $\mu_x(x, y)$ in Example 1. For various combinations of values of **a** and **b** we have drawn Figures **(1 a)** to **(1 f)** by fixing K=100, $10 \le x \le 80$ and $1 \le y \le 5$.
Following are the combinations of **a** and **b** for each figure: **(1 a)**: a=0.1, b=0.9, **(1 b)**: a=0.2,b=0.8, **(1 c)**: a=0.3, b=0.7, **(1 d)**: a=0.7,b=0.3, **(1 e)**: a=0.8,b=0.2, **(1 f)**: a=0.9,b=0.1.

Figure 3

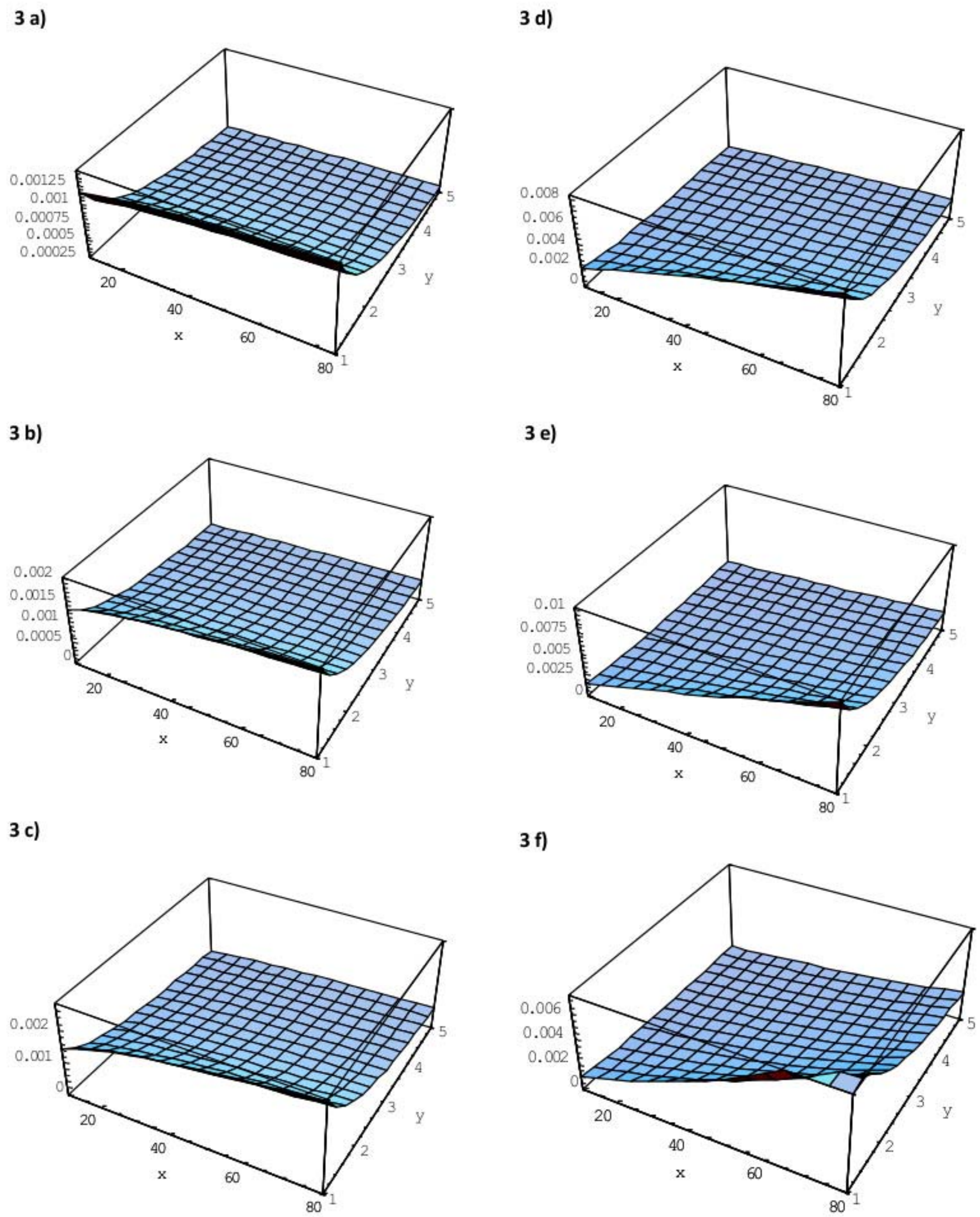

Figure 3. $\mu_y(x, y)$ in Example 1. For various combinations of values of **a** and **b** we have drawn Figures **(1 a)** to **(1 f)** by fixing K=100, $10 \le x \le 80$ and $1 \le y \le 5$.
Following are the combinations of **a** and **b** for each figure: **(1 a)**: a=0.1, b=0.9, **(1 b)**: a=0.2,b=0.8, **(1 c)**: a=0.3, b=0.7, **(1 d)**: a=0.7,b=0.3, **(1 e)**: a=0.8,b=0.2, **(1 f)**: a=0.9,b=0.1.

$$l\left(x_0-\Delta x,y_0-\Delta y\right)=l\left(x_0,y_0\right)-\Delta x\left\{\frac{\partial l}{\partial x}\left(x_0,y_0\right)\right\}-\Delta y\left\{\frac{\partial l}{\partial y}\left(x_0,y_0\right)\right\}+\frac{(\Delta x)^2}{2!}\left\{\frac{\partial^2 l}{\partial x^2}\left(x_0,y_0\right)\right\}$$

$$+\frac{(\Delta y)^2}{2!}\left\{\frac{\partial^2 l}{\partial y^2}\left(x_0,y_0\right)\right\}+\Delta x\Delta y\left\{\frac{\partial^2 l}{\partial x\partial y}\left(x_0,y_0\right)\right\}-\frac{(\Delta x)^3}{3!}\left\{\frac{\partial^3 l}{\partial x^3}\left(x_0,y_0\right)\right\}$$

$$-\frac{(\Delta y)^3}{3!}\left\{\frac{\partial^3 l}{\partial y^3}\left(x_0,y_0\right)\right\}-\frac{\left(\Delta x\right)^2\Delta y}{2}\left\{\frac{\partial^3 l}{\partial x^2\partial y}\left(x_0,y_0\right)\right\}$$

$$-\frac{\Delta x\left(\Delta y\right)^2}{2}\left\{\frac{\partial^3 l}{\partial x\partial y^2}\left(x_0,y_0\right)\right\}+\cdots+\cdots \quad (5)$$

Analysis of first order equations

We try to analyze univariate force of mortality functions by considering higher order derivatives are continuous. Assuming $x \to x_0$, in (2) and (3), we will have $\int_{x_0}^{x}\left((\Delta x)^k l^{(k+1)}(t)/k!\right)dt \to 0.$ Suppose $f^{(n)}(x_0+\Delta x)$ *and* $f^{(n)}(x_0-\Delta x)$ denote Taylor expansion equations when ignoring the $(n+1)^{th}$ derivatives and beyond for $n=2,3,...$ in (2) and (3), then by sequentially ignoring the terms beginning from the term $\frac{(\Delta x)^n}{n!}l^{(n)}(x_0)$ in (2) and (3) for $n=2,3,...$, we will obtain following equations:

$$f^{(1)}(x_0+\Delta x)-f^{(1)}(x_0-\Delta x)=2\Delta x l'(x_0) \quad (6)$$

$$f^{(3)}(x_0+\Delta x)-f^{(3)}(x_0-\Delta x)=2\Delta x l'(x_0)+\frac{2(\Delta x)^3}{3!}l^{(3)}(x_0) \quad (7)$$

$$f^{(5)}(x_0+\Delta x)-f^{(5)}(x_0-\Delta x)=2\Delta x l'(x_0)+\frac{2(\Delta x)^3}{3!}l^{(3)}(x_0)+\frac{2(\Delta x)^5}{5!}l^{(5)}(x_0)$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(2k-1)}(x_0+\Delta x)-f^{(2k-1)}(x_0-\Delta x)=2\Delta x l'(x_0)+\frac{2(\Delta x)^3}{3!}l^{(3)}(x_0)+...+\frac{2(\Delta x)^{(2k-1)}}{(2k-1)!}l^{(2k-1)}(x_0) \quad (8)$$

Therefore,

$$\sum_{j=1}^{k}\left[\left\{f^{(2j+1)}(x_0+\Delta x)-f^{(2j+1)}(x_0-\Delta x)\right\}-\left\{f^{(2j-1)}(x_0+\Delta x)-f^{(2j-1)}(x_0-\Delta x)\right\}\right]=$$

$$\frac{2(\Delta x)^3}{3!}l^{(3)}(x_0)+\frac{2(\Delta x)^5}{5!}l^{(5)}(x_0)+...+\frac{2(\Delta x)^{(2k+1)}}{(2k+1)!}l^{(2k+1)}(x_0) \quad (9)$$

Hence we will obtain,

$$\sum_{j=1}^{k}\left[\left\{f^{(2j+1)}(x_0+\Delta x)-f^{(2j+1)}(x_0-\Delta x)\right\}-\left\{f^{(2j-1)}(x_0+\Delta x)-f^{(2j-1)}(x_0-\Delta x)\right\}\right]=$$
$$\left\{f^{(2k+1)}(x_0+\Delta x)-f^{(2k+1)}(x_0-\Delta x)\right\}-\left\{f^{(1)}(x_0+\Delta x)-f^{(1)}(x_0-\Delta x)\right\} \quad (10)$$

Using the relation $l^{(1)}(x_0)=\frac{1}{2\Delta x}\left[f^{(1)}(x_0+\Delta x)-f^{(1)}(x_0-\Delta x)\right]$, we write,

$$\left.\frac{dl(x)}{dx}\right|_{x=x_0}=\frac{1}{2\Delta x}[\left\{f^{(2k+1)}(x_0+\Delta x)-f^{(2k+1)}(x_0-\Delta x)\right\}-$$

$$\sum_{j=1}^{k}\Big[\big\{f^{(2j+1)}(x_0+\Delta x)-f^{(2j+1)}(x_0-\Delta x)\big\}-\big\{f^{(2j-1)}(x_0+\Delta x)-f^{(2j-1)}(x_0-\Delta x)\big\}\Big]\Bigg] \tag{11}$$

Dividing the (6) by the term $2\Delta x$ on both the sides and integrating it from age x to $x+m$, we obtain,

$$\int_x^{x+m}\frac{\{f^{(1)}(y_0+\Delta y)-f^{(1)}(y_0-\Delta y)\}}{2\Delta y}dy=\int_x^{x+m}\frac{d}{dy}l(y)\Big|_{y=y_0}dy$$

$$=-\int_x^{x+m}-\frac{1}{l(y)}\frac{d}{dy}l(y)\Big|_{y=y_0} l(y)\big|_{y=y_0}dy=-\int_x^{x+m}\mu(y)\big|_{y=y_0} l(y)\big|_{y=y_0}dy \tag{12}$$

where, $\mu(x)$, the force of mortality function, which is defined as $\big(-1/l(x)\big)(d/dx)l(x).$

Dividing the (7) by the term $2\Delta x$ on both the sides and integrating it from age x to $x+m$, we obtain,

$$\int_x^{x+m}\left[\frac{\{f^{(3)}(y_0+\Delta y)-f^{(3)}(y_0-\Delta y)\}}{2\Delta y}-\frac{(\Delta y)^2}{3!}l^{(3)}(y_0)\right]dy=-\int_x^{x+m}\mu(y)\big|_{y=y_0} l(y)\big|_{y=y_0}dy \tag{13}$$

Now, multiplying $l(x)\big|_{x=x_0}$ and $(-1/l(x))\big|_{x=x_0}$ to the ((11)), and integrating between ages x and $x+m$, we will obtain,

$$-\int_x^{x+m}-\frac{1}{l(y)}\frac{d}{dy}l(y)\Big|_{y=y_0} l(y)\big|_{y=y_0}dy=\int_x^{x+m}-\frac{1}{2\Delta y}\big[\big\{f^{(2k+1)}(y_0+\Delta y)-f^{(2k+1)}(y_0-\Delta y)\big\}$$

$$-\sum_{j=1}^{k}\Big[\big\{f^{(2j+1)}(y_0+\Delta y)-f^{(2j+1)}(y_0-\Delta y)\big\}-\big\{f^{(2j-1)}(y_0+\Delta y)-f^{(2j-1)}(y_0-\Delta y)\big\}\Big]\Big]$$

$$=-\int_x^{x+m}\frac{d}{dy}l(y)\Big|_{y=y_0}dy=ml'(y_0) \tag{14}$$

If there are any deaths during the age $(x_0+\Delta x)$ to $(x_0-\Delta x)$, then $f^{(1)}(x_0+\Delta x)<f^{(1)}(x_0-\Delta x).$ In the absence of deaths, we have $f^{(1)}(x_0+\Delta x)=f^{(1)}(x_0-\Delta x)$ and $\frac{dl(x)}{dx}\Big|_{x=x_0}<0.$ This argument is true for difference of other higher order expressions.

Analysis of second order equations

Suppose $g^{(n)}(x_0+\Delta x,y_0+\Delta x)\ and\ g^{(n)}(x_0-\Delta x,y_0-\Delta y)$ denote equations when ignoring the terms from $(n+1)^{th}$ partial derivatives and beyond for $n=2,3,...$ in (4) and (5), then by sequentially ignoring the terms beginning from the n^{th} order partial derivative terms in (4) and (5) for $n=2,3,...$, we will obtain following equations:

$$g^{(1)}(x_0+\Delta x,y_0+\Delta y)-g^{(1)}(x_0-\Delta x,y_0-\Delta y)=2\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+2\Delta y\left\{\frac{\partial l}{\partial y}(x_0,y_0)\right\}$$

$$g^{(3)}(x_0+\Delta x,y_0+\Delta y)-g^{(3)}(x_0-\Delta x,y_0-\Delta y)=2\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+2\Delta y\left\{\frac{\partial l}{\partial y}(x_0,y_0)\right\}$$

$$+\frac{(\Delta x)^3}{3!}\left\{\frac{\partial^3 l}{\partial x^3}(x_0,y_0)\right\}+\frac{(\Delta y)^3}{3!}\left\{\frac{\partial^3 l}{\partial y^3}(x_0,y_0)\right\}$$

$$+\frac{(\Delta x)^2 \Delta y}{2}\left\{\frac{\partial^3 l}{\partial x^2 \partial y}(x_0,y_0)\right\}+\frac{\Delta x(\Delta y)^2}{2}\left\{\frac{\partial^3 l}{\partial x \partial y^2}(x_0,y_0)\right\}$$

$$\vdots \qquad \vdots$$

$$g^{(2k-1)}(x_0+\Delta x, y_0+\Delta y)-g^{(2k-1)}(x_0-\Delta x, y_0-\Delta y)=2\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+2\Delta y\left\{\frac{\partial l}{\partial y}(x_0,y_0)\right\}+\ldots$$

$$+\frac{2(\Delta x)^{2k-j-1}(\Delta y)^j}{(2k-1)!}\left[\sum_{j=0}^{2k-1}\binom{2k-1}{j}\times\left\{\frac{\partial^{(2k-1)} l}{\partial x^{(2k-j-1)}\partial y^j}(x_0,y_0)\right\}\right]$$

$$\vdots \qquad \vdots$$

Therefore,

$$\sum_{j=1}^{k}\Big[\left\{g^{(2k+1)}(x_0+\Delta x, y_0+\Delta y)-g^{(2k+1)}(x_0-\Delta x, y_0-\Delta y)\right\}$$

$$-\left\{g^{(2k-1)}(x_0+\Delta x, y_0+\Delta y)-g^{(2k-1)}(x_0-\Delta x, y_0-\Delta y)\right\}\Big]=\frac{(\Delta x)^3}{3!}\left\{\frac{\partial^3 l}{\partial x^3}(x_0,y_0)\right\}$$

$$+\frac{(\Delta y)^3}{3!}\left\{\frac{\partial^3 l}{\partial y^3}(x_0,y_0)\right\}+\frac{(\Delta x)^2\Delta y}{2}\left\{\frac{\partial^3 l}{\partial x^2\partial y}(x_0,y_0)\right\}+\frac{\Delta x(\Delta y)^2}{2}\left\{\frac{\partial^3 l}{\partial x\partial y^2}(x_0,y_0)\right\}+\ldots..$$

$$\cdots+\frac{2(\Delta x)^{2k-j+1}(\Delta y)^j}{(2k+1)!}\left[\sum_{j=0}^{2k+1}\binom{2k+1}{j}\left\{\frac{\partial^{(2k+1)} l}{\partial x^{(2k-j+1)}\partial y^j}(x_0,y_0)\right\}\right] \tag{15}$$

Force of mortality for the two variables (x,y) is evaluated at the point (x_0,y_0) using partial derivatives as follows:

$$\frac{\partial \mu(x,y)}{\partial x}=\lim_{\Delta x\to 0}\frac{\mu(x_0+\Delta x,y_0)-\mu(x_0,y_0)}{\Delta x} \tag{16}$$

$$\frac{\partial \mu(x,y)}{\partial y}=\lim_{\Delta y\to 0}\frac{\mu(x_0,y_0+\Delta y)-\mu(x_0,y_0)}{\Delta y} \tag{17}$$

where, we define,

$$\mu(x_0+\Delta x,y_0)=-\frac{1}{l(x_0+\Delta x,y_0)}\frac{\partial l(x_0+\Delta x,y_0)}{\partial x} \tag{18}$$

$$\mu(x_0,y_0+\Delta y)=-\frac{1}{l(x_0,y_0+\Delta y)}\frac{\partial l(x_0,y_0+\Delta y)}{\partial y} \tag{19}$$

$$\mu_x(x_0,y_0)=-\frac{1}{l(x_0,y_0)}\frac{\partial l(x_0,y_0)}{\partial x}\ \text{ for eq.(16)} \tag{20}$$

$$\mu_y(x_0,y_0)=-\frac{1}{l(x_0,y_0)}\frac{\partial l(x_0,y_0)}{\partial y}\ \text{ for eq.(17)} \tag{21}$$

Using (4) and as $\Delta y\to 0$, we obtain,

$$l(x_0+\Delta x,y_0)=l(x_0,y_0)+\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+\frac{(\Delta x)^2}{2!}\left\{\frac{\partial^2 l}{\partial x^2}(x_0,y_0)\right\}+\frac{(\Delta x)^3}{3!}\left\{\frac{\partial^3 l}{\partial x^3}(x_0,y_0)\right\} \tag{22}$$

Using (5) and as $\Delta x\to 0$, we obtain,

$$l(x_0,y_0+\Delta y)=l(x_0,y_0)+\Delta y\left\{\frac{\partial l}{\partial y}(x_0,y_0)\right\}+\frac{(\Delta y)^2}{2!}\left\{\frac{\partial^2 l}{\partial y^2}(x_0,y_0)\right\}+\frac{(\Delta y)^3}{3!}\left\{\frac{\partial^3 l(x_0,y_0)}{\partial y^3}\right\} \tag{23}$$

Therefore,

$$\frac{\partial l}{\partial x}(x_0+\Delta x,y_0)=\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+\Delta x\left\{\frac{\partial^2 l}{\partial x^2}(x_0,y_0)\right\}+\frac{(\Delta x)^2}{2!}\left\{\frac{\partial^3 l}{\partial x^3}(x_0,y_0)\right\}+\frac{(\Delta x)^3}{3!}\left\{\frac{\partial^4 l}{\partial x^4}(x_0,y_0)\right\} \tag{24}$$

$$\frac{\partial l}{\partial y}(x_0,y_0+\Delta y)=\left\{\frac{\partial l}{\partial y}(x_0,y_0)\right\}+\Delta y\left\{\frac{\partial^2 l}{\partial y^2}(x_0,y_0)\right\}+\frac{(\Delta y)^2}{2!}\left\{\frac{\partial^3 l}{\partial y^3}(x_0,y_0)\right\}+\frac{(\Delta y)^3}{3!}\left\{\frac{\partial^4 l}{\partial y^4}(x_0,y_0)\right\} \tag{25}$$

Let us now derive the equation of the type (15) with the conditions $\Delta y\to 0$ and $\Delta x\to 0$ and extending these equations up to the general term. Suppose, $\Delta y\to 0$ in (4) and (5) , then

$$l(x_0+\Delta x,y_0)=l(x_0,y_0)+\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+\frac{(\Delta x)^2}{2!}\left\{\frac{\partial^2 l}{\partial x^2}(x_0,y_0)\right\}+\cdots$$
$$+\frac{(\Delta x)^n}{n!}\left\{\frac{\partial^n l}{\partial x^n}(x_0,y_0)\right\}+\cdots \tag{26}$$

$$l(x_0-\Delta x,y_0)=l(x_0,y_0)-\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+\frac{(\Delta x)^2}{2!}\left\{\frac{\partial^2 l}{\partial x^2}(x_0,y_0)\right\}-\cdots$$
$$+(-1)^n\frac{(\Delta x)^n}{n!}\left\{\frac{\partial^n l}{\partial x^n}(x_0,y_0)\right\}+\cdots \tag{27}$$

and $\Delta x\to 0$ in (4) and (5), then

$$l(x_0,y_0+\Delta y)=l(x_0,y_0)+\Delta y\left\{\frac{\partial l}{\partial y}(x_0,y_0)\right\}+\frac{(\Delta y)^2}{2!}\left\{\frac{\partial^2 l}{\partial y^2}(x_0,y_0)\right\}+\cdots$$
$$+\frac{(\Delta y)^n}{n!}\left\{\frac{\partial^n l}{\partial y^n}(x_0,y_0)\right\}+\cdots \tag{28}$$

$$l(x_0,y_0-\Delta y)=l(x_0,y_0)-\Delta y\left\{\frac{\partial l}{\partial y}(x_0,y_0)\right\}+\frac{(\Delta y)^2}{2!}\left\{\frac{\partial^2 l}{\partial y^2}(x_0,y_0)\right\}-\cdots$$
$$+(-1)^n\frac{(\Delta y)^n}{n!}\left\{\frac{\partial^n l}{\partial y^n}(x_0,y_0)\right\}+\cdots \tag{29}$$

Sequentially, ignoring the n^{th} order terms from (26) and (27), and denoting these new equations as $g_1^{(n)}(x_0+\Delta x,y_0)$ and $g_1^{(n)}(x_0-\Delta x,y_0)$ for $n=2,3,\ldots$, we obtain below set of equations.

$$g_1^{(1)}(x_0+\Delta x,y_0)-g_1^{(1)}(x_0-\Delta x,y_0)=2\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\} \tag{30}$$

$$g_1^{(3)}(x_0+\Delta x,y_0)-g_1^{(3)}(x_0-\Delta x,y_0)=2\Delta x\left\{\frac{\partial l}{\partial x}(x_0,y_0)\right\}+\frac{2(\Delta x)^3}{3!}\left\{\frac{\partial^3 l}{\partial x^3}(x_0,y_0)\right\}$$

$$\vdots \qquad \vdots$$

$$g_1^{(2k-1)}\left(x_0+\Delta x,y_0\right)-g_1^{(2k-1)}\left(x_0-\Delta x,y_0\right)=2\Delta x\left\{\frac{\partial l}{\partial x}\left(x_0,y_0\right)\right\}+\frac{2(\Delta x)^3}{3!}\left\{\frac{\partial^3 l}{\partial x^3}\left(x_0,y_0\right)\right\}+....$$

$$\cdots+\frac{2(\Delta x)^{(2k-1)}}{(2k-1)!}\left\{\frac{\partial^{(2k-1)}l}{\partial x^{(2k-1)}}\left(x_0,y_0\right)\right\}$$

Therefore,

$$\sum_{j=0}^{k}\left[\left\{g_1^{(2j+1)}\left(x_0+\Delta x,y_0\right)-g_1^{(2j+1)}\left(x_0-\Delta x,y_0\right)\right\}-\left\{g_1^{(2j-1)}\left(x_0+\Delta x,y_0\right)-g_1^{(2j-1)}\left(x_0-\Delta x,y_0\right)\right\}\right]=$$

$$\sum_{j=1}^{k}\frac{2(\Delta x)^{(2j+1)}}{(2j+1)!}\left\{\frac{\partial^{(2j+1)}l}{\partial x^{(2j+1)}}\left(x_0,y_0\right)\right\} \tag{31}$$

$$=\left\{g_1^{(2k+1)}\left(x_0+\Delta x,y_0\right)-g_1^{(2k+1)}\left(x_0-\Delta x,y_0\right)\right\}-\left\{g_1^{(1)}\left(x_0+\Delta x,y_0\right)-g_1^{(1)}\left(x_0-\Delta x,y_0\right)\right\}$$

Sequentially, ignoring the n^{th} order terms from (28) and (29), and denoting these new equations as $g_2^{(n)}\left(x_0,y_0+\Delta y\right)$ and $g_2^{(n)}\left(x_0,y_0+\Delta y\right)$ for $n=2,3,...$, we obtain below set of equations.

$$g_2^{(1)}\left(x_0,y_0+\Delta y\right)-g_2^{(1)}\left(x_0,y_0-\Delta y\right)=2\Delta y\left\{\frac{\partial l}{\partial y}\left(x_0,y_0\right)\right\} \tag{32}$$

$$g_2^{(3)}\left(x_0,y_0+\Delta y\right)-g_2^{(3)}\left(x_0,y_0-\Delta y\right)=2\Delta y\left\{\frac{\partial l}{\partial y}\left(x_0,y_0\right)\right\}+\frac{2(\Delta y)^3}{3!}\left\{\frac{\partial^3 l}{\partial y^3}\left(x_0,y_0\right)\right\}$$

$$\vdots\vdots \qquad \vdots\vdots$$

$$g_2^{(2k-1)}\left(x_0,y_0+\Delta y\right)-g_2^{(2k-1)}\left(x_0,y_0-\Delta y\right)=2\Delta y\left\{\frac{\partial l}{\partial y}\left(x_0,y_0\right)\right\}+\frac{2(\Delta y)^3}{3!}\left\{\frac{\partial^3 l}{\partial y^3}\left(x_0,y_0\right)\right\}+....$$

$$\cdots+\frac{2(\Delta y)^{(2k-1)}}{(2k-1)!}\left\{\frac{\partial^{(2k-1)}l}{\partial y^{(2k-1)}}\left(x_0,y_0\right)\right\}$$

Therefore,

$$\sum_{j=0}^{k}\left[\left\{g_2^{(2j+1)}\left(x_0,y_0+\Delta y\right)-g_2^{(2j+1)}\left(x_0,y_0+\Delta y\right)\right\}-\left\{g_2^{(2j-1)}\left(x_0,y_0+\Delta y\right)-g_2^{(2j-1)}\left(x_0,y_0+\Delta y\right)\right\}\right]=$$

$$\sum_{j=1}^{k}\frac{2(\Delta y)^{(2j+1)}}{(2j+1)!}\left\{\frac{\partial^{(2j+1)}l}{\partial y^{(2j+1)}}\left(x_0,y_0\right)\right\} \tag{33}$$

$$=\left\{l^{(2k+1)}\left(x_0,y_0+\Delta y\right)-l^{(2k+1)}\left(x_0,y_0+\Delta y\right)\right\}-\left\{l^{(1)}\left(x_0,y_0+\Delta y\right)-l^{(1)}\left(x_0,y_0-\Delta y\right)\right\}$$

Now substituting the (22) and (24) in the (18), we get

$$\mu\left(x_0+\Delta x,y_0\right)=-\frac{1}{\left[l\left(x_0,y_0\right)+\sum_{j=1}^{\infty}\frac{(\Delta x)^j}{j!}\left\{\frac{\partial^j l}{\partial x^j}\left(x_0,y_0\right)\right\}\right]}\sum_{j=0}^{\infty}\frac{(\Delta x)^j}{j!}\left\{\frac{\partial^{j+1}l}{\partial x^{j+1}}\left(x_0,y_0\right)\right\} \tag{34}$$

and, substituting the (23) and (25) in the (19), we get

$$\mu\left(x_0,y_0+\Delta y\right)=-\frac{1}{\left[l\left(x_0,y_0\right)+\sum_{j=1}^{\infty}\frac{(\Delta y)^j}{j!}\left\{\frac{\partial^j l}{\partial y^j}\left(x_0,y_0\right)\right\}\right]}\sum_{j=0}^{\infty}\frac{(\Delta y)^j}{j!}\left\{\frac{\partial^{j+1}l}{\partial y^{j+1}}\left(x_0,y_0\right)\right\} \tag{35}$$

Using (30) and (32), we obtain forces of mortalities for two variables as follows:

$$\mu\left(x_{0,}y_0\right) = -\frac{\left\{g_1^{(1)}\left(x_0+\Delta x,y_0\right)-g_1^{(1)}\left(x_0-\Delta x,y_0\right)\right\}}{2\Delta x l\left(x_0,y_0\right)}$$

$$\mu\left(x_{0,}y_0\right) = -\frac{\left\{g_2^{(1)}\left(x_0,y_0+\Delta y\right)-g_2^{(1)}\left(x_0,y_0-\Delta y\right)\right\}}{2\Delta y l\left(x_0,y_0\right)}$$

Hence the derivative forces of mortalities are as follows:

$$\frac{\partial\mu\left(x_,y\right)}{\partial x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{\left\{g_1^{(1)}\left(x_0+\Delta x,y_0\right)-g_1^{(1)}\left(x_0-\Delta x,y_0\right)\right\}}{2\Delta x l\left(x_0,y_0\right)}-\frac{\sum_{j=0}^{\infty}\frac{(\Delta x)^j}{j!}\left\{\frac{\partial^{j+1}l}{\partial x^{j+1}}\left(x_0,y_0\right)\right\}}{\left[l\left(x_0,y_0\right)+\sum_{j=1}^{\infty}\frac{(\Delta x)^j}{j!}\frac{\partial^j l}{\partial x^j}\left(x_0,y_0\right)\right]}\right] \tag{36}$$

$$\frac{\partial\mu\left(x_,y\right)}{\partial y} = \lim_{\Delta y\to 0}\frac{1}{\Delta y}\left[\frac{\left\{g_2^{(1)}\left(x_0,y_0+\Delta y\right)-g_2^{(1)}\left(x_0,y_0-\Delta y\right)\right\}}{2\Delta y l\left(x_0,y_0\right)}-\frac{\sum_{j=0}^{\infty}\frac{(\Delta y)^j}{j!}\left\{\frac{\partial^{j+1}l}{\partial y^{j+1}}\left(x_0,y_0\right)\right\}}{\left[l\left(x_0,y_0\right)+\sum_{j=1}^{\infty}\frac{(\Delta y)^j}{j!}\left\{\frac{\partial^j l}{\partial y^j}\left(x_0,y_0\right)\right\}\right]}\right] \tag{37}$$

Analysis of third order equations

Suppose s be the function number of survivors at age x with two more influencing variables y and z, then the three variable survival function $s\left(x,y,z\right)$ evaluated at $\left\{x_o,y_0,z_0\right\}$ can be written as

$$\begin{aligned}s\left(x_0+\Delta x,y_0+\Delta y,z_0+\Delta z\right) &= s\left(x_0,y_0,z_0\right)+\Delta x\left\{\frac{\partial s}{\partial x}\left(x_0,y_0,z_0\right)\right\}+\Delta y\left\{\frac{\partial s}{\partial y}\left(x_0,y_0,z_0\right)\right\}\\ &+\Delta z\left\{\frac{\partial s}{\partial z}\left(x_0,y_0,z_0\right)\right\}+\frac{\Delta x^2}{2}\left\{\frac{\partial^2 s}{\partial x^2}\left(x_0,y_0,z_0\right)\right\}\\ &+\frac{\Delta y^2}{2}\left\{\frac{\partial^2 s}{\partial y^2}\left(x_0,y_0,z_0\right)\right\}+\frac{\Delta z^2}{2}\left\{\frac{\partial^2 s}{\partial z^2}\left(x_0,y_0,z_0\right)\right\}\\ &+\Delta x\Delta y\left\{\frac{\partial^2 s}{\partial x\partial y}\left(x_0,y_0,z_0\right)\right\}+\Delta x\Delta z\left\{\frac{\partial^2 s}{\partial x\partial z}\left(x_0,y_0,z_0\right)\right\}+\Delta y\Delta z\left\{\frac{\partial^2 s}{\partial y\partial z}\left(x_0,y_0,z_0\right)\right\}+\ldots..\\ &\cdots+\left[\sum_{n_1,n_2,n_3}\frac{1}{n_1!n_2!n_3!}\frac{\partial^k s}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3}}\Delta x^{n_1}\Delta y^{n_2}\Delta z^{n_3}\right]\end{aligned} \tag{38}$$

Similarly,

$$\begin{aligned}s\left(x_0-\Delta x,y_0-\Delta y,z_0-\Delta z\right) &= s\left(x_0,y_0,z_0\right)-\Delta x\left\{\frac{\partial s}{\partial x}\left(x_0,y_0,z_0\right)\right\}-\Delta y\left\{\frac{\partial s}{\partial y}\left(x_0,y_0,z_0\right)\right\}\\ &-\Delta z\left\{\frac{\partial s}{\partial z}\left(x_0,y_0,z_0\right)\right\}+\frac{\Delta x^2}{2}\left\{\frac{\partial^2 s}{\partial x^2}\left(x_0,y_0,z_0\right)\right\}+\frac{\Delta y^2}{2}\left\{\frac{\partial^2 s}{\partial y^2}\left(x_0,y_0,z_0\right)\right\}+\frac{\Delta z^2}{2}\left\{\frac{\partial^2 s}{\partial z^2}\left(x_0,y_0,z_0\right)\right\}\\ &+\Delta x\Delta y\left\{\frac{\partial^2 s}{\partial x\partial y}\left(x_0,y_0,z_0\right)\right\}+\Delta x\Delta z\left\{\frac{\partial^2 s}{\partial x\partial z}\left(x_0,y_0,z_0\right)\right\}+\Delta y\Delta z\left\{\frac{\partial^2 s}{\partial y\partial z}\left(x_0,y_0,z_0\right)\right\}+\ldots..\end{aligned}$$

$$\cdots + (-1)^k \left[\sum_{n_1,n_2,n_3} \frac{1}{n_1!n_2!n_3!} \frac{\partial^k s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3} \right] \tag{39}$$

Here $k = n_1 + n_2 + n_3$.

Let $h^{(n)}\left(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z\right)$ and $h^{(n)}\left(x_0-\Delta x, y_0-\Delta y, z_0-\Delta z\right)$ denote equations after ignoring the terms with derivatives beginning from the $(n+1)^{th}$ order ($n = 1,2,...$) in the (38) and (39). We will obtain following equations:

$$\left.\begin{array}{l} h^{(1)}\left(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z\right) \\ -h^{(1)}\left(x_0-\Delta x, y_0-\Delta y, z_0-\Delta z\right) \end{array}\right\} = 2\Delta x \left\{ \frac{\partial s}{\partial x}\left(x_0, y_0, z_0\right) \right\} + 2\Delta y \left\{ \frac{\partial s}{\partial y}\left(x_0, y_0, z_0\right) \right\}$$

$$+2\Delta z \left\{ \frac{\partial s}{\partial z}\left(x_0, y_0, z_0\right) \right\}$$

$$\vdots \qquad \vdots$$

$$\left.\begin{array}{l} h^{(2k-1)}\left(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z\right) \\ -h^{(2k-1)}\left(x_0-\Delta x, y_0-\Delta y, z_0-\Delta z\right) \end{array}\right\} = 2\Delta x \left\{ \frac{\partial s}{\partial x}\left(x_0, y_0, z_0\right) \right\} + 2\Delta y \left\{ \frac{\partial s}{\partial y}\left(x_0, y_0, z_0\right) \right\}$$

$$+2\Delta z \left\{ \frac{\partial s}{\partial z}\left(x_0, y_0, z_0\right) \right\} + \cdots + \sum_{n_1,n_2,n_3} \frac{2}{n_1!n_2!n_3!} \frac{\partial^{(2k-1)} s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3}$$

$$\left(here\ 2k-1 = n_1+n_2+n_3\right)$$

$$\vdots \qquad \vdots$$

Therefore,

$$\sum_{j=1}^{k} \Big[\left\{ h^{(2k+1)}\left(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z\right) - h^{(2k+1)}\left(x_0-\Delta x, y_0-\Delta y, z_0-\Delta z\right) \right\}$$

$$-\left\{ g^{(2k-1)}\left(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z\right) - g^{(2k-1)}\left(x_0-\Delta x, y_0-\Delta y, z_0+\Delta z\right) \right\} \Big] =$$

$$\sum_{n_1,n_2,n_3} \frac{2}{n_1!n_2!n_3!} \frac{\partial^3 s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3} \qquad \left(here\ 3 = n_1+n_2+n_3\right)$$

$$= + \cdots + \sum_{n_1,n_2,n_3} \frac{2}{n_1!n_2!n_3!} \frac{\partial^{(2k+1)} s}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} \Delta x^{n_1} \Delta y^{n_2} \Delta z^{n_3}$$

$$\left(here\ 2k+1 = n_1+n_2+n_3\right)$$

We define following three force of mortality functions evaluated at $\left(x_0, y_0, z_0\right)$:

$$\mu_x\left(x_0, y_0, z_0\right) = -\frac{1}{s\left(x_0, y_0, z_0\right)} \frac{\partial s\left(x_0, y_0, z_0\right)}{\partial x}$$

$$\mu_y\left(x_0, y_0, z_0\right) = -\frac{1}{s\left(x_0, y_0, z_0\right)} \frac{\partial s\left(x_0, y_0, z_0\right)}{\partial y}$$

$$\mu_z\left(x_0, y_0, z_0\right) = -\frac{1}{s\left(x_0, y_0, z_0\right)} \frac{\partial s\left(x_0, y_0, z_0\right)}{\partial z}$$

and further we define three functions of forces of mortality as follows:

$$\mu\left(x_0+\Delta x, y_0, z_0\right) = -\frac{1}{s\left(x_0+\Delta x, y_0, z_0\right)} \frac{\partial s\left(x_0+\Delta x, y_0, z_0\right)}{\partial x}$$

$$\mu\left(x_0, y_0+\Delta y, z_0\right) = -\frac{1}{s\left(x_0, y_0+\Delta y, z_0\right)} \frac{\partial s\left(x_0, y_0+\Delta y, z_0\right)}{\partial y}$$

$$\mu\left(x_0,y_0,z_0+\Delta z\right)=-\frac{1}{s\left(x_0,y_0,z_0+\Delta z\right)}\frac{\partial s\left(x_0,y_0,z_0+\Delta z\right)}{\partial z}$$

Using the above definitions, we obtain following rates evaluated at $\left(x_0,y_0,z_0\right)$:

$$\frac{\partial\mu\left(x,y,z\right)}{\partial x}=\lim_{\Delta x\to 0}\frac{\mu\left(x_0+\Delta x,y_0,z_0\right)-\mu\left(x_0,y_0,z_0\right)}{\Delta x}$$
$$\frac{\partial\mu\left(x,y,z\right)}{\partial y}=\lim_{\Delta y\to 0}\frac{\mu\left(x_0,y_0+\Delta y,z_0\right)-\mu\left(x_0,y_0,z_0\right)}{\Delta y}$$
$$\frac{\partial\mu\left(x,y,z\right)}{\partial z}=\lim_{\Delta z\to 0}\frac{\mu\left(x_0,y_0,z_0+\Delta z\right)-\mu\left(x_0,y_0,z_0\right)}{\Delta z}$$

By taking pairs of limits $(\Delta y\to 0,\ \Delta z\to 0)$, $(\Delta x\to 0,\ \Delta z\to 0)$, and $(\Delta x\to 0,\ \Delta y\to 0)$, separately in the (38), we obtain following three equations:

$$s\left(x_0+\Delta x,y_0,z_0\right)=s\left(x_0,y_0,z_0\right)+\Delta x\left\{\frac{\partial s}{\partial x}\left(x_0,y_0,z_0\right)\right\}+\cdots+\frac{\Delta x^n}{n!}\left\{\frac{\partial^n s}{\partial x^n}\left(x_0,y_0,z_0\right)\right\}+\ldots.. \quad (40)$$

$$s\left(x_0,y_0+\Delta y,z_0\right)=s\left(x_0,y_0,z_0\right)+\Delta y\left\{\frac{\partial s}{\partial y}\left(x_0,y_0,z_0\right)\right\}+\cdots+\frac{\Delta y^n}{n!}\left\{\frac{\partial^n s}{\partial y^n}\left(x_0,y_0,z_0\right)\right\}+\cdots \quad (41)$$

$$s\left(x_0,y_0,z_0+\Delta z\right)=s\left(x_0,y_0,z_0\right)+\Delta z\left\{\frac{\partial s}{\partial z}\left(x_0,y_0,z_0\right)\right\}+\cdots+\frac{\Delta z^n}{n!}\left\{\frac{\partial^n s}{\partial z^n}\left(x_0,y_0,z_0\right)\right\}+\cdots \quad (42)$$

By taking pairs of limits $(\Delta y\to 0,\ \Delta z\to 0)$, $(\Delta x\to 0,\ \Delta z\to 0)$, and $(\Delta x\to 0,\ \Delta y\to 0)$, separately in the (39), we obtain following three equations:

$$s\left(x_0-\Delta x,y_0,z_0\right)=s\left(x_0,y_0,z_0\right)-\Delta x\left\{\frac{\partial s}{\partial x}\left(x_0,y_0,z_0\right)\right\}+\cdots+(-1)^n\frac{\Delta x^n}{n!}\left\{\frac{\partial^n s}{\partial x^n}\left(x_0,y_0,z_0\right)\right\}+\cdots \quad (43)$$

$$s\left(x_0,y_0-\Delta y,z_0\right)=s\left(x_0,y_0,z_0\right)-\Delta y\left\{\frac{\partial s}{\partial y}\left(x_0,y_0,z_0\right)\right\}+\cdots+(-1)^n\frac{\Delta y^n}{n!}\left\{\frac{\partial^n s}{\partial y^n}\left(x_0,y_0,z_0\right)\right\}+\cdots \quad (44)$$

$$s\left(x_0,y_0,z_0-\Delta z\right)=s\left(x_0,y_0,z_0\right)-\Delta z\left\{\frac{\partial s}{\partial z}\left(x_0,y_0,z_0\right)\right\}+\cdots+(-1)^n\frac{\Delta z^n}{n!}\left\{\frac{\partial^n s}{\partial z^n}\left(x_0,y_0,z_0\right)\right\}+\cdots \quad (45)$$

Suppose $h_1^n\left(x_0+\Delta x,y_0,z_0\right)$, $h_1^n\left(x_0-\Delta x,y_0,z_0\right)$; $h_2^n\left(x_0,y_0+\Delta y,z_0\right)$, $h_2^n\left(x_0,y_0-\Delta y,z_0\right)$ and $h_3^n\left(x_0,y_0,z_0+\Delta z\right)$, $h_3^n\left(x_0,y_0,z_0-\Delta z\right)$ for $n=1,2,3,\ldots$ denote the functions by ignoring the terms from the order $(n+1)$ in the pairs of equations (40), (43); (41), (44) and (42), (45), then we will obtain following three series of sequences of difference functions:

$$h_1^{(1)}\left(x_0+\Delta x,y_0,z_0\right)-h_1^{(1)}\left(x_0-\Delta x,y_0,z_0\right)=2\Delta x\left\{\frac{\partial s}{\partial x}\left(x_0,y_0,z_0\right)\right\}$$

$$h_1^{(3)}\left(x_0+\Delta x,y_0,z_0\right)-h_1^{(3)}\left(x_0-\Delta x,y_0,z_0\right)=2\Delta x\left\{\frac{\partial s}{\partial x}\left(x_0,y_0,z_0\right)\right\}+\frac{2\Delta x^3}{3!}\left\{\frac{\partial^3 s}{\partial x^3}\left(x_0,y_0,z_0\right)\right\}$$

$$\vdots\vdots \qquad \vdots\vdots$$

$$h_1^{(2k-1)}\left(x_0+\Delta x,y_0,z_0\right)-h_1^{(2k-1)}\left(x_0-\Delta x,y_0,z_0\right)=2\Delta x\left\{\frac{\partial s}{\partial x}\left(x_0,y_0,z_0\right)\right\}+\frac{2\Delta x^3}{3!}\left\{\frac{\partial^3 s}{\partial x^3}\left(x_0,y_0,z_0\right)\right\}+$$

$$\cdots+\frac{2\Delta x^{(2k-1)}}{(2k-1)!}\left\{\frac{\partial^{(2k-1)}s}{\partial x^{(2k-1)}}(x_0,y_0,z_0)\right\}$$

$$\vdots \qquad \vdots$$

Therefore,

$$\sum_{j=1}^{k}\begin{bmatrix}\left\{h_1^{(2j+1)}(x_0+\Delta x,y_0,z_0)-h_1^{(2j+1)}(x_0-\Delta x,y_0,z_0)\right\}- \\ \left\{h_1^{(2j-1)}(x_0+\Delta x,y_0,z_0)-h_1^{(2j-1)}(x_0-\Delta x,y_0,z_0)\right\}\end{bmatrix}=\frac{2\Delta x^3}{3!}\left\{\frac{\partial^3 s}{\partial x^3}(x_0,y_0,z_0)\right\}+$$

$$\cdots+\frac{2\Delta x^{(2k+1)}}{(2k+1)!}\left\{\frac{\partial^{(2k+1)}s}{\partial x^{(2k+1)}}(x_0,y_0,z_0)\right\} \tag{46}$$

$$h_2^{(1)}(x_0,y_0+\Delta y,z_0)-h_2^{(1)}(x_0,y_0-\Delta y,z_0)=2\Delta y\left\{\frac{\partial s}{\partial y}(x_0,y_0,z_0)\right\}$$

$$h_2^{(3)}(x_0,y_0+\Delta y,z_0)-h_2^{(3)}(x_0,y_0-\Delta y,z_0)=2\Delta y\left\{\frac{\partial s}{\partial y}(x_0,y_0,z_0)\right\}+\frac{2\Delta y^3}{3!}\left\{\frac{\partial^3 s}{\partial y^3}(x_0,y_0,z_0)\right\}$$

$$\vdots \qquad \vdots$$

$$h_2^{(2k-1)}(x_0,y_0+\Delta y,z_0)-h_2^{(2k-1)}(x_0,y_0-\Delta y,z_0)=2\Delta y\left\{\frac{\partial s}{\partial y}(x_0,y_0,z_0)\right\}+\frac{2\Delta y^3}{3!}\left\{\frac{\partial^3 s}{\partial y^3}(x_0,y_0,z_0)\right\}$$

$$\cdots+\frac{2\Delta y^{2k-1}}{(2k-1)!}\left\{\frac{\partial^{2k-1}s}{\partial y^{2k-1}}(x_0,y_0,z_0)\right\}$$

$$\vdots$$

Therefore,

$$\sum_{j=1}^{k}\begin{bmatrix}\left\{h_2^{(2j+1)}(x_0,y_0+\Delta y,z_0)-h_2^{(2j+1)}(x_0,y_0-\Delta y,z_0)\right\}- \\ \left\{h_2^{(2j-1)}(x_0,y_0+\Delta y,z_0)-h_2^{(2j-1)}(x_0,y_0-\Delta y,z_0)\right\}\end{bmatrix}=\frac{2\Delta y^3}{3!}\left\{\frac{\partial^3 s}{\partial y^3}(x_0,y_0,z_0)\right\}+$$

$$\cdots+\frac{2\Delta y^{2k+1}}{(2k+1)!}\left\{\frac{\partial^{2k+1}s}{\partial y^{2k+1}}(x_0,y_0,z_0)\right\} \tag{47}$$

$$h_3^{(1)}(x_0,y_0,z_0+\Delta z)-h_3^{(1)}(x_0,y_0,z_0-\Delta z)=2\Delta z\left\{\frac{\partial s}{\partial z}(x_0,y_0,z_0)\right\}$$

$$h_3^{(3)}(x_0,y_0,z_0+\Delta z)-h_3^{(3)}(x_0,y_0,z_0-\Delta z)=2\Delta z\left\{\frac{\partial s}{\partial z}(x_0,y_0,z_0)\right\}+\frac{2\Delta z^3}{3!}\left\{\frac{\partial^3 s}{\partial z^3}(x_0,y_0,z_0)\right\}$$

$$\vdots \qquad \vdots$$

$$h_3^{(2k-1)}(x_0,y_0,z_0+\Delta z)-h_3^{(2k-1)}(x_0,y_0,z_0-\Delta z)=2\Delta z\left\{\frac{\partial s}{\partial z}(x_0,y_0,z_0)\right\}+\frac{2\Delta z^3}{3!}\left\{\frac{\partial^3 s}{\partial z^3}(x_0,y_0,z_0)\right\}$$

$$\cdots+\frac{2\Delta z^{(2k-1)}}{(2k-1)!}\left\{\frac{\partial^{(2k-1)}s}{\partial z^{(2k-1)}}(x_0,y_0,z_0)\right\}$$

$$\vdots \qquad \vdots$$

Therefore,

$$\sum_{j=1}^{k}\begin{bmatrix}\left\{h_3^{(2j+1)}(x_0,y_0,z_0+\Delta z)-h_3^{(2j+1)}(x_0,y_0,z_0-\Delta z)\right\}- \\ \left\{h_3^{(2j-1)}(x_0,y_0,z_0+\Delta z)-h_3^{(2j-1)}(x_0,y_0,z_0-\Delta z)\right\}\end{bmatrix}=\frac{2\Delta z^3}{3!}\left\{\frac{\partial^3 s}{\partial z^3}(x_0,y_0,z_0)\right\}+$$

$$\cdots+\frac{2\Delta z^{(2k+1)}}{(2k+1)!}\left\{\frac{\partial^{(2k+1)}s}{\partial z^{(2k+1)}}(x_0,y_0,z_0)\right\} \tag{48}$$

The rate of changes in the survival function with respect to one variable and corresponding forces of mortalities for three variables can be obtained using the following derivations.

$$\frac{\partial s(x_0+\Delta x,y_0,z_0)}{\partial x}=\frac{\partial s(x_0,y_0,z_0)}{\partial x}+\Delta x\left\{\frac{\partial^2 s}{\partial x^2}(x_0,y_0,z_0)\right\}+\cdots+\frac{\Delta x^n}{n!}\left\{\frac{\partial^{n+1}s}{\partial x^{n+1}}(x_0,y_0,z_0)\right\}+\cdots$$

$$\frac{\partial s(x_0,y_0+\Delta y,z_0)}{\partial y}=\frac{\partial s(x_0,y_0,z_0)}{\partial y}+\Delta y\left\{\frac{\partial^2 s}{\partial y^2}(x_0,y_0,z_0)\right\}+\cdots+\frac{\Delta y^n}{n!}\left\{\frac{\partial^{n+1}s}{\partial y^{n+1}}(x_0,y_0,z_0)\right\}+\cdots$$

$$\frac{\partial s(x_0,y_0,z_0+\Delta z)}{\partial z}=\frac{\partial s(x_0,y_0,z_0)}{\partial z}+\Delta z\left\{\frac{\partial^2 s}{\partial z^2}(x_0,y_0,z_0)\right\}+\cdots+\frac{\Delta z^n}{n!}\left\{\frac{\partial^{n+1}s}{\partial z^{n+1}}(x_0,y_0,z_0)\right\}+\cdots$$

$$\mu(x_0+\Delta x,y_0,z_0)=-\frac{1}{s(x_0,y_0,z_0)+\sum_{i=1}^{\infty}\left[\frac{\Delta x^i}{i!}\left\{\frac{\partial^i s}{\partial x^i}(x_0,y_0,z_0)\right\}\right]}\sum_{i=0}^{\infty}\left[\frac{\Delta x^i}{i!}\left\{\frac{\partial^{i+1}s}{\partial x^{i+1}}(x_0,y_0,z_0)\right\}\right]$$

$$\mu(x_0,y_0+\Delta y,z_0)=-\frac{1}{s(x_0,y_0,z_0)+\sum_{i=1}^{\infty}\left[\frac{\Delta y^i}{i!}\left\{\frac{\partial^i s}{\partial y^i}(x_0,y_0,z_0)\right\}\right]}\sum_{i=0}^{\infty}\left[\frac{\Delta y^i}{i!}\left\{\frac{\partial^{i+1}s}{\partial y^{i+1}}(x_0,y_0,z_0)\right\}\right]$$

$$\mu(x_0,y_0,z_0+\Delta z)=-\frac{1}{s(x_0,y_0,z_0)+\sum_{i=1}^{\infty}\left[\frac{\Delta z^i}{i!}\left\{\frac{\partial^i s}{\partial z^i}(x_0,y_0,z_0)\right\}\right]}\sum_{i=0}^{\infty}\left[\frac{\Delta z^i}{i!}\left\{\frac{\partial^{i+1}s}{\partial z^{i+1}}(x_0,y_0,z_0)\right\}\right]$$

$$\frac{\partial\mu(x,y,z)}{\partial x}=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{1}{s(x_0,y_0,z_0)}\left\{\frac{h_1^{(1)}(x_0+\Delta x,y_0,z_0)-h_1^{(1)}(x_0-\Delta x,y_0,z_0)}{2\Delta x}\right\}\right.$$
$$\left.-\left\{\frac{\sum_{i=0}^{\infty}\left[\frac{\Delta x^i}{i!}\left\{\frac{\partial^{i+1}s}{\partial x^{i+1}}(x_0,y_0,z_0)\right\}\right]}{s(x_0,y_0,z_0)+\sum_{i=1}^{\infty}\left[\frac{\Delta x^i}{i!}\left\{\frac{\partial^i s}{\partial x^i}(x_0,y_0,z_0)\right\}\right]}\right\}\right]$$

$$\frac{\partial\mu(x,y,z)}{\partial y}=\lim_{\Delta y\to 0}\frac{1}{\Delta y}\left[\frac{1}{s(x_0,y_0,z_0)}\left\{\frac{h_y^{(1)}(x_0,y_0+\Delta y,z_0)-h_2^{(1)}(x_0,y_0-\Delta y,z_0)}{2\Delta y}\right\}\right.$$
$$\left.-\left\{\frac{\sum_{i=0}^{\infty}\left[\frac{\Delta y^i}{i!}\left\{\frac{\partial^{i+1}s}{\partial y^{i+1}}(x_0,y_0,z_0)\right\}\right]}{s(x_0,y_0,z_0)+\sum_{i=1}^{\infty}\left[\frac{\Delta y^i}{i!}\left\{\frac{\partial^i s}{\partial y^i}(x_0,y_0,z_0)\right\}\right]}\right\}\right]$$

$$\frac{\partial\mu(x,y,z)}{\partial z}=\lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[\frac{1}{s(x_0,y_0,z_0)}\left\{\frac{h_3^{(1)}(x_0,y_0,z_0+\Delta z)-h_3^{(1)}(x_0,y_0,z_0-\Delta z)}{2\Delta z}\right\}\right.$$

$$-\left\{\frac{\sum_{i=0}^{\infty}\left[\frac{\Delta y^{i}}{i!}\left\{\frac{\partial^{i+1}s}{\partial y^{i+1}}\left(x_0,y_0,z_0\right)\right\}\right]}{s\left(x_0,y_0,z_0\right)+\sum_{i=1}^{\infty}\left[\frac{\Delta y^{i}}{i!}\left\{\frac{\partial^{i}s}{\partial y^{i}}\left(x_0,y_0,z_0\right)\right\}\right]}\right\}\Bigg]\Bigg]$$

Conclusions

Our numerical examples and analytical derivations does encourage to validate results obtained by univariate force of mortality with that of bivariate and multivariate forces of mortality functions. Majority of the mortality data analysed consider age as a predominant variable (Tuljapurkar et al., 2000; Bebbington et al. 2007; Finkelstein, 2005; Gavrilov et al., 1991) in forecasting and analysis. Some insect populations as well age is considered as a predominant variable in mortality analysis (Wilmoth, 1998; Carey et al., 1992). Mortality data analysed does indicate that considering only one variable in concluding the causes of decline could lead to incomplete hypothesis, thus warrants further analysis.

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