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Research Article

Stochastic Models for Human Fertility

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Abstract

This paper is basically based on the "George Simmons Memorial Lecture" delivered at the 35th annual conference of IASP held at Rohtak (2013). It gives a brief description of various stochastic models for human fertility specially developed at the Department of Statistics, Banaras Hindu University, Varanasi, India. Although it does not give a comprehensive list of all the papers related to stochastic models for human fertility but provides a description of various directions in which the models have been developed. The present paper also includes a description some recent papers published after the above conference.

Introduction

Human reproduction is a very complex process. Although it is basically a biological process but it is very much influenced by various social, economic, cultural and behavioral factors so much so that in many situations these factors become more prominent than the biological factors. Hence for any clear-cut understanding of the human fertility behavior one should have a deep understanding of the above two aspects of fertility. Researchers have used various approaches to study human fertility from different angles according to their interest. One of these approaches, although not very common, has been the use of mathematical models to study the human fertility behaviour.

By a model, we mean any conceptualization of real situation. If this is done in terms of mathematical relationship(s) then the model may be called as a mathematical model corresponding to a real phenomenon under study. It is a theoretical structure, a logico-deductive conceptualized scheme, expressed through a single equation or a set of equations meant to unfold process of the phenomenon under consideration. These models have novel interpretative, predictive and communicative values which enhance their utility considerably. Depending upon the nature of the various aspects of the phenomenon under investigation, the mathematical model fall broadly in either of two categories

- Deterministic
- Stochastic (i.e. Probabilistic)

The deterministic model has an element of certainty in the outcome and in a situation where the end result is certain and almost predictable, the deterministic model can serve the purpose well. On the other hand, in a situation where the phenomenon is of random nature and the outcome is uncertain, a stochastic model is appropriate. While the mean or expected values are given by the stochastic model may agree with that of the deterministic model based on sufficiently similar assumptions, the stochastic model offers more insight into the unforeseen consequences otherwise difficult to understand in a complicated phenomenon.

Uses of Mathematical Models

- Model builder generally makes certain assumptions about the system based on his experience and intuition and then tries to describe the behavior of the system in terms of mathematical equation(s). All these help in understanding the phenomenon in a better fashion.

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- One of the difficulties in demographic research is its non-experimental nature. Many times a researcher finds more than a single factor operating over a phenomenon under study. Since all the analyses, in this case, are based on human beings and hence it is difficult to control a number of factors at the same time and repeat the experiment under identical conditions. It is also essential to know what would happen by changing a factor if other factors are kept constant. It is possible to achieve this through models. Moreover, a researcher may also be interested in knowing the interrelationships among different factors. This can also be accomplished through models.

- At a later stage, the researcher may utilize the model to estimate some of the parameters (unknowns) which may appear in its equations by obtaining a fit to the observed data.

- Sometimes a researcher may find some results which may appear inconsistent but when the problem is tackled with the help of a model, the apparent inconsistency may be explained.

The Qualities of a Model Builder

- A comprehensive knowledge of the phenomenon under consideration for which a model is to be developed.

- A comprehensive mathematical and statistical skill to put the phenomenon in the form of mathematical relations which are realistic to the phenomenon and have interpretative, predictive values with possibilities of estimation of unknown characteristics (parameters) inherent in the phenomenon.

In fact model building is an art as well as science. This can be well understood if we remember the following statement of Prof. R. P. Agnew in his book “Differential Equation" (McGraw-Hill, New York, 1942, p. 30). Prof. Agnew wrote, “*much of the progress in science is due to men who have the courage to make assumptions, the good sense to make reasonable assumptions, and the ability to draw correct conclusions from the assumptions*".

Thus if a researcher wants to develop any mathematical model for human fertility, the researcher should have a clear-cut understanding of the human reproduction process. Fertility is basically measured by live birth which is an outcome of a successful termination of a conception. A live birth is the complete expulsion or extraction from its mother of a product of conception, irrespective of the duration of pregnancy, which, after such separation, breathes or shows any other evidence of life, such as beating of the heart, pulsation of the umbilical cord, or any definite movement of voluntary muscles, whether or not the umbilical cord has been cut or the placenta is attached.

For live birth, normally the following three events have to occur.

1) There should be an intercourse.
2) The intercourse should result into a conception.
3) The conception should result into a live birth.

Keeping this into consideration, American sociologists Davis and Blake (1956) first identified a list of intermediate variables which may influence the fertility. The list of 11 intermediate variables is given below:

I. Factors affecting exposure to intercourse

A. Those governing the formation and dissolution of unions in the reproductive period
1. Age of entry into sexual unions
2. Permanent celibacy; proportion of women never entering sexual unions
3. Amount of reproductive period spent after or between unions
   a. When unions are broken by divorce, separation or desertion
   b. When unions are broken by death of husband

B. Those governing the exposure to intercourse within unions
4. Voluntary abstinence
5. Involuntary abstinence (from impotence, illness, and unavoidable but temporary separations)
6. Coital frequency (excluding periods of abstinence)

II. Factors affecting exposure to conception
7. Fecundity or infecundity, as affected by involuntary causes
8. Use or non-use of contraception
a. By mechanical and chemical means
b. By other means
9. Fecundity or infecundity, as affected by voluntary causes (sterilization, subincision, medical treatment, etc.)

III. Factors affecting gestation and successful parturition
10. Foetal mortality from involuntary causes
11. Foetal mortality from voluntary causes

The biological and behavioral factors through which social, economic and environmental factors affect fertility have been called as 'Intermediate variables of fertility'. The primary characteristic of intermediate fertility variables is their direct influence on fertility. If any intermediate variable, such as the prevalence of contraception changes, then fertility changes necessarily also (assuming the other intermediate fertility variables remaining constant) while this is not necessarily the case for an indirect determinant such as income or education (Bongaarts, 1978).

Although Davis and Blake (1956) tried to give an extensive list of intermediate variables (factors affecting fertility directly) but they did not provide any suitable methodology to measure the impact of various factors quantitatively which may provide more meaningful conclusions. Alternatively, other researchers tried to develop measures to quantify the different aspects of human fertility.

Broadly these quantitative measures can be specified as:

(1) Fecundability
(2) The probability that a conception results in a live birth
(3) Non-susceptible period associated with a conception/birth
(4) The probability that a female is fecund (or sterile) at various stages of her reproductive life

We now try to explain these in brief:

(1) **Fecundability:** Before defining fecundability, we must recognize that fertility is measured by live births and live births are successful termination of conceptions into live births. In fact occurrence of a conception to a female is a random event. Biologically a conception can take place during a menstrual cycle only if:

a) The menstrual cycle is ovulatory i.e. an ovum is released from the ovary during the menstrual cycle.

b) An intercourse takes place during the fertile period of the menstrual cycle. Although no precise length of the fertile period is known for a female but normally it occurs in the middle of the menstrual cycle and its length is about 3-4 days.

c) The intercourse during the fertile period leads to the fertilization of the ovum i.e. occurrence of a conception.

It is also known that even if an intercourse occurs during the fertile period, there is no surety that conception will occur definitely in the menstrual cycle. Since the fertile period is not known precisely, the intercourses occur somewhat randomly over the cycle and there is no surety of occurrence of conception even in presence of intercourse during the fertile period, the occurrence of conception during a menstrual cycle becomes an event of random nature and consequently, it has a probability associated with it. In the light of above facts, the fecundability is defined as the probability that a non-contracepting, susceptible female will conceive in a menstrual cycle.

It is to be further mentioned that the length of a menstrual cycle is precisely not constant but roughly it is around one month. Consequently, the time interval of a menstrual cycle is generally taken as one month. Thus practically fecundability is considered as a probability of conception in one month. Thus in modelling, the unit of time is taken for one month and in any time interval, time is considered to be discrete taking values 0,1,2,3,… ('0' representing the initial time).

In many studies, instead of treating time to be discrete, it is considered as continuous. For this situation, we consider the probability of conception during an infinitely small interval $(t, t+\Delta t)$ as $\lambda(t).\Delta t+O(\Delta t)$. Here $\lambda(t)$ is considered as conception rate at time 't'. Many times, $\lambda(t)$ is considered as independent of 't' i.e. λ (say), then λ is called as conception rate (per unit of time). If the unit of time is taken for one month, then λ is almost equivalent to fecundability. However, if the unit of time is taken as one year, then λ is called conception rate per year or yearly conception rate. Both the situations i.e. treating time to be discrete or continuous have extensively been considered by researchers according to their methodologies. Normally in literature, fecundability has been denoted by 'p' whereas λ (or 'm') has been denoted for the conception rate.

(2) **The probability of a conception leading to a live birth:** It is known that normally after a gestation of about nine months, the conception terminates into a live birth. (Sometimes multiple births may also occur such as twins or more). However, in many situations a conception may not terminate into a live birth, instead, it may terminate into an abortion or stillbirth (a birth where the child is dead before the birth). Thus it is not necessary that every conception will result in a live birth. Consequently, we consider a probability say θ which denote the probability that a conception results into a live birth. Consequently, $(1-\theta)$ is taken as the probability that the conception will not result in a live birth. Some researchers have used the term incomplete birth for the abortion/stillbirth. For simplicity, researchers have also assumed one to one correspondence between conception and live birth i.e. θ to be equal to one.

(3) **The non-susceptible period associated with a conception:** It is known that after the occurrence of conception, it will terminate either in the form of live birth or abortion or stillbirth after a gestation period. During this period, there is no possibility of any other conception. It is also known that after the termination of pregnancy, there is no menstruation for certain period and in this period also, there is no possibility of a conception. The period from the termination of pregnancy till the occurrence of first menstruation after termination of pregnancy is known as post-partum amenorrhea period. The sum of the gestation period and the post-partum amenorrhea period has been termed as non-susceptible period associated with the conception. Practically, in the non-susceptible period, the probability of conception is zero. At this stage, one point needs special mention. In fact, the ovulation takes place before menstruation and if the ovum is not fertilized then the menstruation occurs. However, if the ovum is fertilized then no menstruation occurs and the conception takes place which terminates into a live birth or abortion/stillbirth after the completion of the gestation period. Thus in this situation, practically no menstruation occurs before the next birth/conception. In this situation, the duration from the termination of pregnancy to the occurrence of next conception may be treated as a non-susceptible period. Normally the non-susceptible period is denoted by 'h'.

(4) **The probability that a female is fecund (or sterile) at various stages of her reproductive life:** Fecundity refers to the physiological capacity of a female to conceive. Thus a female is said to be fecund if she has the capacity to conceive. The absence of fecundity is considered as 'sterility'. Technically, every female is non-fecund before menarche and after menopause. Further, the fertility performance of a non-fecund female will definitely be zero but a female with zero fertility may not necessarily be sterile (non-fecund). Further, a female may be fertile for some period and then becomes sterile, for example, after sterilization. Thus the sterility is usually classified as primary sterility or secondary sterility. Obviously, sterility plays a major role in the fertility study. In literature, usually, α denotes the probability that the female is fecund.

In this study we have discussed following types of stochastic models mainly developed at Department of Statistics, Banaras Hindu University:

- Models for number of births/conceptions in a given period
- Models for different types of birth interval
- Estimation of Parity Progression Ratios (PPR) and Instantaneous Parity Progression Ratios (IPPR) from Open and Closed birth interval data
- Son preference and sex ratio at birth

The descriptions of above models are given below:

Models for number of births and conceptions: Dandekar (1955) proposed a stochastic model for number of births in a given period and tried to apply it to real data collected in a survey in two districts in Maharashtra. The data mainly refer to number of births given by females in a given age group. In this model the unit of time was firstly considered as one year. It is said that initially Dandekar tried to fit the model with the help of binomial distribution considering one-year period of reproduction as a trial assuming that in a year, a female gives a birth or does not give the birth. Giving birth in a year may be considered as success whereas not giving birth as failure (assuming that the probability of giving more than one birth in a year is very low and hence it can be ignored). He also thought that the probability of success will remain same from trial to trial (year to year) and hence thought that binomial distribution may be an appropriate model for the number of births in the given period. However, when he applied the model to the real data, he could not get a satisfactory fit of the model to the data. Later on, with discussion from other experts, it was noted that the condition of independence of trials in binomial distribution is not satisfied with the given situation because it was noted that if a female gives birth in a particular year, then the probability of giving another birth in the next year will be somewhat low. Based on this thought, Dandekar modified his binomial model in the form of modified binomial distribution. The modification was mainly done by incorporating that if a female gives a birth at certain point of time then for certain period of time (non-susceptible period) the probability of giving birth is zero. He also derived the modified Poisson distribution as limiting form of binomial distribution. This paper is considered as the backbone of modeling the human fertility in terms of number of births in a given period.

Although the paper of Dandekar was published but even the modified form of binomial distribution did not give a satisfactorily good fit of the model to the data. While doing his Ph.D, at University of California, Dr. S.N. Singh was given a problem to examine the reasons by his supervisor Dr. J. Neyman that why modified binomial distribution by Dandekar was not providing a satisfactory fit. Initially for some time he could not visualize any reasonable answer to the problem. After sometime, he noticed from his experience of his village life that some of the females are basically primarily sterile and are not able to give any birth even during their whole reproductive period. In the light of above fact, he modified the model proposed by Dandekar incorporating that a certain proportion of females will not be able to give any birth during the period of five or seven years. Incorporating this fact he published two papers in the Journal of American Statistical Association (Singh 1963, Singh 1968). In (1963) paper, unit of time was taken as discrete (one month) while in (1968) paper the time was treated as continuous.

The models proposed by Dandekar (1955) and Singh (1963, 1968) were based on a common assumption that there is one to one correspondence between conception and birth (i.e. ignoring foetal loss). Later on, Singh and Bhattacharya (1970) published a paper on a number of conceptions in a given period incorporating the possibility of foetal loss. In 1971, Singh & Bhattacharya (1971) published another paper for a number of births and conceptions in a given period of time in the bivariate form.

In all the above papers, it has been assumed that the level of fecundability (or conception rate) throughout the period of observation is constant. This may be reasonably true if the period of observation is not large. However, it is also a recognized fact that fecundability may vary according to age/parity. Incorporating this fact, a parity dependent model for number of births in a given period was proposed by Singh et al (1974). The model was applied to the data collected in "Demographic survey of Varanasi Rural 1969-70" and the fit was found to be reasonably good. The model also provided the consequential impact of various hypothetical family planning programmes on fertility in a given period which is one of the important uses of mathematical models. In the above model, possibility of variation in fecundability and sterility over parity was considered. Later on Bhattcharya et al. (1984) incorporated the possibility of foetal loss also and applied it to compute births averted under different hypothetical family planning programmes. Sheps et al. (1973) have also given an extensive description of models for number of conceptions and births.

The above-mentioned models have mainly been derived for a period (0, T) where '0' represents the time of marriage or start of the sexual union. So the distributions are useful only for this situation. However, if the start of the observational period is not marriage but a distant point since marriage, then the models need modification. Dandekar (1955) have termed this situation as an abrupt sequence of time. Singh and Yadava (1977) considered this situation as equilibrium birth process and derived a model for number of conceptions in a given period (T, T+t) where T is a distant point since marriage.

In all the above papers, it has been assumed that the female is exposed to the risk of conception for the whole period (except during non-susceptible period). However, there may be situations where the female is not exposed to the risk of conception even when she is not in the non-susceptible period. This may be quite common for the migrated couples where the male partner is away from home for a prolonged period and the female partner is living at home. In this situation, the male partner normally visits the home only for a short period say 1-2 months in a year. Singh et al. (1981), have proposed a model for number of births in a given period for such type of couples and given a procedure to estimate the fecundability level of such couples. After application of the model to data of "Demographic survey of Varanasi Rural 1969-70", it was found that the fecundability level of such couples was substantially large in comparison to the fecundability level of females living with their husbands in the village. This difference in the level of fecundability of two types of couples was mainly attributed to the higher coital frequency of migrated couples when the male partner visited home.

Sometimes models are required to describe the distribution of births to a female during a specified period given that she has experienced a birth at the end of the observational period. Such situations may arise for estimating fertility parameters utilizing data collected from females coming to hospitals for delivery. Singh et al. (1975) have propounded a probability distribution for number of births to a female during the period (T, T+t) given that she has experienced a birth at time T+t where T is a distant point since marriage.

There is a property of a uniform number of births during a period (T, T+t) in equilibrium birth process. A natural question arises whether the births for females giving a specified number of births in time (T, T+t) are also uniformly distributed over the period (T, T+t) Yadava and Srivastava (1993) developed a methodology to investigate the above issue. They found that the births for this situation are not uniformly distributed over time although there was a symmetry of specific type for births in different segments of period 'h' where 'h' is the non-susceptible period associated with a birth. This issue was mainly investigated to find the probability distribution of number of surviving children at time T+t out of births in the period (T, T+t). Later on, Yadava and Tiwari (2007) extended the idea of this behaviour by considering one-year segments of the period (T, T+t).

Models for birth intervals: Before describing various stochastic models on birth intervals, it is desirable to explain various types of birth intervals.

a) **First Birth Interval:** The interval between marriage to first birth is defined as a first birth interval.

b) **Closed Birth Interval or Inter-live Birth Interval:** The interval between two successive live births is known as closed birth interval or inter-live birth interval.

c) **Last Closed Birth Interval:** It is the interval between the last and next to last live births of a female. For example, if a female has given four births before the survey date then her last closed birth interval is the time interval between her third and fourth birth.

d) **Interior Birth Interval:** It is the closed birth interval that begins and ends in any segment of time (which may be the age-group or the marital duration or interval between two survey dates).

e) **Open Birth Interval:** The interval between the survey date and the date of last birth is referred as an open birth interval.

f) **Forward Birth Interval:** The interval between survey date and the date of next live birth posterior to the survey date is known as a forward birth interval.

g) **Straddling Birth Interval:** An interval is said to be straddling at a particular age or at a particular time point if one birth occurs before that age or specified point of time and the other birth occurs later. Any closed birth interval that straddles the survey date is said to be straddling birth interval.

Normally it is supposed that an inverse relationship between number of births and birth interval exists, so researchers have also tried to study fertility through the use of birth intervals. Initially, the researchers tried to examine the distribution of time of first conception /birth from marriage. Singh (1964) proposed a model for the time of first birth and applied it to the data of a survey near Varanasi. Almost at the same time Potter and Parker (1964), Sheps (1964, 1967), Pathak (1967) have derived models for first conceptive delay under various sets of assumptions. Perhaps the choice for the first conceptive delay was considered for simplicity because this was not going to involve the non-susceptible period. However, such models were giving an estimate of fecundability level at the early part of marriage duration.

In all the above models, the basic assumption was that the time of the first conception follows a geometric distribution/exponential distribution. The choice of geometric distribution was purely based on the concept of geometric distribution derived from a sequence of Bernoulli trials. We know that for Bernoulli trials, the following three conditions have to be satisfied.

1. For each trial, there are only two possibilities, one being called a success and the other being called a failure.
2. The probability of success p remains constant from trial to trial.
3. The trials are independent.

While developing the model for the time of the first conception, the authors considered one menstrual cycle (one month) as a trial and it was considered that in each trial (in each menstrual cycle) there can be a conception or there cannot be a conception. On the other hand, it was also assumed that the probability of conception in a month i.e. fecundability remains constant from month to month till the occurrence of the first conception. Further, it was assumed safely that the trials are independent i.e. outcome of a trial does not affect the outcome of the other trials at least till the time of the first conception. Since exponential distribution is an analog of geometric distribution if time is considered as continuous, hence some authors have also assumed the exponential distribution for the time of the first conception.

Srinivasan (1966) developed a model to describe the distribution of time between successive live births incorporating the possibility of foetal losses as well as variation in post-partum amenorrhea period among females. Such intervals have been called as closed birth intervals. He also applied his proposed model to the data of Gandhigram Survey. The data on inter-live birth interval have been considered as providing the estimate of fecundability between two consecutive births which may be useful to study variation in fecundability over parity. He also introduced the idea of the open birth interval which is similar to the backward recurrence time in renewal theory.

Then came a classical paper by Sheps et al. (1970) which thoroughly discussed the truncation effect on birth intervals and suggested that due care is taken before analyzing birth interval data to draw valid conclusions. Almost, at the same time, Wolfers (1968) also discussed the issue of analyzing closed birth interval by distinguishing between the mean birth interval (births) and mean birth interval (women) and gave examples to show the difference between the two. Later on, Menkan and Sheps (1972) and Sheps et al.(1973) added another dimension known as the sampling frame. This dimension typically describes the difficulties of the impact of sampling frame on the distribution of birth intervals. Sampling frame (ascertainment plan) for a birth interval refers to the manner in which a birth interval is determined. Apart from other considerations, the ascertainment plan or sampling frame influences the distribution of birth interval so much so that many times one may draw incorrect or misleading conclusions.

We explain the problem of sampling frame with an example given below: Suppose a researcher is interested in the study of interval between first birth and second birth based on data collected in a survey on birth intervals of females who are in their reproductive age group (say females of 15-49 years) at the time of survey. Now, the females under consideration may be

1) Females who have given exactly two births before the survey date and the data on duration between first and second births are considered.

2) Females who have given at least two births before the time of survey and data on the interval between first and second births are considered.

3) Females whose marital duration is more than fifteen years are considered and the data on duration between first and second births are considered.

It can be seen that in all these cases, data relate to time between first and second births but their distributions will be entirely different, even though the data are from the same population.

The problem associated with the analysis of birth interval data can also be seen with the classical example of the **Waiting Time Paradox** [see Feller (1966)]. Although the description of waiting time paradox is given quite long back, it seems desirable to give a brief description of the paradox here also which gives an indication of problems of analysis of data on duration variables of which birth interval is a special case.

Suppose buses come at random at a bus stop. The time interval (X) between arrivals of consecutive buses follows an exponential distribution with p.d.f.

$$f(x) = \lambda e^{-\lambda x} x > 0, \qquad \lambda > 0$$

Obviously $E(X) = \frac{1}{\lambda}$

If a person reaches the bus stop at random, what will be his average waiting time to get the next bus? Answer to this question can be given in two ways.

(1) The average time between consecutive buses is $\frac{1}{\lambda}$ and the person reaches the bus stop at random. So on an average the person will be reaching the bus stop mid-way between the two buses. The average waiting time for the person for the next bus will be half of $\frac{1}{\lambda}$ i.e. $\frac{1}{2\lambda}$.

(2) Exponential distribution has the property of ‘memory loss’ or it ‘forgets the past’. Consequently, whenever the person reaches the bus stop, his average waiting time for the next bus will be $\frac{1}{\lambda}$.

This is precisely the ‘waiting time paradox’. On one argument the average time will be $\frac{1}{2\lambda}$ while on the other argument it is $\frac{1}{\lambda}$. Solution to this problem was given utilizing results of distribution theory. In fact actual value of average waiting time is $\frac{1}{\lambda}$ but the other argument is also not completely incorrect. In fact by putting the condition that the person reaches the bus stop, the inter arrival time between two such consecutive buses will not remain exponential. It can be shown that this distribution has the p.d.f.

$$f(x) = \lambda^2 x e^{-\lambda x}$$

which has mean $\frac{2}{\lambda}$ and half of this becomes $\frac{1}{\lambda}$ which is the average waiting time for the next bus. Precisely, above gives the basis for treating straddling birth interval as different from usual closed birth interval.

Although such result was in existence quite long back, in another context, Srinivasan (1968) did not make any distinction between such straddling birth intervals from the usual closed birth interval and in fact, he assumed that the mean of the open birth interval will be half of the mean of the closed birth interval. In fact, open birth interval is similar to backward recurrence time whereas waiting time for next bus may be considered as forward recurrence time in the context of renewal theory in the stochastic process. Later on, Leridon (1969) pointed out about the fallacy of result in Srinivasan (1968) and gave a correct expression for a mean of the open birth interval.

Besides sampling frame and truncation effect, heterogeneity with respect to parameters involved in a model may also alter the distribution significantly. Hence, the selection effect within a heterogeneous population is also an important concern. Heterogeneity in the underlying population places difficulties in the way of interpretation of all statistical data based on averages. In fact the problem discussed by Wolfers(1968) is a problem associated with heterogeneity among females with respect to their fertility parameters. The problem of analysis of duration of post-partum amenorrhea discussed in Singh et al. (1979b) is also problem associated with heterogeneity which has been termed as **Selection Bias.**

The above issue can be explained with the following example. Suppose we consider a group of recently married females and follow them to ascertain the time of their first birth. Suppose X denotes the time of first conception. If we assume that fecundability p remains constant from month to month and further assume that X follows a geometric distribution with p.m.f.

$$P[X = x] = pq^{x-1} \quad , x=1,2,3,\ldots, 0<p<1$$

then we see that probability of conception in the first month will be p while it will be qp in the second month $(q = 1 - p)$. Thus the conditional probability of conception in the second month given that female has not conceived in the first month is $\frac{qp}{1-p} = p$. Similarly, it can be shown that conditional probability of conception in a month given that she has not conceived earlier will always be p. However, if we assume that the population of females is heterogeneous with respect to fecundability with p.d.f. $f(p)$, then the probability of conception in first month will be

$$\bar{p} = \int pf(p)dp$$

The probability of conception in the second month will be equal to $\int qpf(p)dp$ which can be written as $\int pf(p)dp - \int p^2 f(p)dp$.

But $\int p^2 f(p)dp = \bar{p}^2 + {\sigma_p}^2$ where ${\sigma_p}^2$ is the variance of p, so the probability of conception in the second month will be equal to$\bar{p} - \bar{p}^2 - {\sigma_p}^2$.

Therefore, the conditional probability of conception in second month given that the female has not conceived in the first month will be $\frac{\bar{p}-\bar{p}^2-{\sigma_p}^2}{1-\bar{p}} = \bar{p} - \frac{{\sigma_p}^2}{1-p}$ which is less than $\bar{p}$ (since ${\sigma_p}^2 > 0$).

Similarly, the conditional probability of conception in any month will be declining over time. We may draw two inferences from this finding. Either we can say that the fecundability is decreasing over time or we can say that the population under consideration is heterogeneous with respect to fecundability. However, it does not seem reasonable to assume that fecundability is declining over time so the second conclusion seems to be more reasonable.

Although forward birth interval and straddling birth interval have little practical use because these require the conduct of another survey after a sufficiently large period from the previous survey, still their theoretical utility is never minimized because the straddling birth interval provides a theoretical basis to distinguish it from usual closed birth interval and the forward birth interval as different from half of the closed birth interval. In this context, Singh et al.(1978) proposed a model for forward birth interval whereas Yadava and Pandey (1989); Yadava and Srivastava (1994) developed models for straddling birth interval. Similarly, Singh and Bhattacharya (1986) proposed a model for an interior birth interval which may be more practical if the period of observation is even short.

Although Wolfers (1968) made a distinction between the mean birth interval (births) and mean birth interval (women) and gave a methodology to establish interrelationship between the two but his argument for the relationship was quite heuristic. Later on, Singh et al.(1989) proposed a more rigorous approach to find the relationship between the two and pointed out certain inconsistencies in the results of Wolfers (1968).

In all the models discussed above, the fertility parameters during the birth interval period have been assumed to be constant. However, it has been seen that fecundability/conception rate is a bit low in the early part of the menstruating interval because of reduced coitus rate when the child is too young. For this situation, Bhattachaya et al. (1988) proposed models for closed birth interval assuming fecundability to be time-dependent and discussed related problems of analysis of data. Later on Yadava et al. (2009) also assumed the conception rate to be time-dependent by assuming that the conception rate after the start of menstruating interval linearly increases up to certain time and then becomes constant till the occurrence of next conception and applied it to a data set obtained in a survey conducted in a hospital.

Till now we mainly discussed about closed birth intervals. However, almost at the same time, the idea of the open birth interval as a random segment of the closed birth interval was also introduced by Srinivasan (1968) although a little attempt was made to obtain specific distributions of the open birth interval. At this stage, Pathak (1970) derived a model for open birth interval for females with specific marital duration and specific parity. He showed that the proposed model was quite insensitive to the changes in conception rate which implied that data on open birth interval corresponding to fixed parity and marital duration is almost of no use in drawing conclusions on conception rate. Later on Singh et al. (1982) proposed a parity dependent model for open birth interval and found that it is also less sensitive to change in conception rate unless it changes drastically from parity to parity. However, it was seen that open birth interval irrespective of parity is quite sensitive to change in fertility parameters (Singh et al.(1979a).

The models discussed above for closed birth intervals are mainly for a closed birth interval of a specific order. However, Yadava and Sharma (2007) made an analysis of consecutive closed birth intervals and found expression for correlation between the two assuming that the population is heterogeneous with respect to the non-susceptible period but for a female it is constant in the two consecutive intervals. Applying this methodology to a real data set, they found that the two components of a closed birth interval viz. PPA and menstruating interval are negatively correlated showing that if PPA is small, couples try to delay the menstruating interval, perhaps by reducing their coital frequency at least for some time. However, this may not be the case if the PPA is large.

Later, Kumar and Yadava (2015) obtained an expression for correlation between consecutive closed birth intervals assuming heterogeneity in conception rate among females. They found that the correlation coefficient between the consecutive intervals is positive in a heterogeneous population. However, it is important to mention here that Yadava and Sharma (2007) found the correlation coefficient between PPA and menstruating interval using a specific data set while Kumar and Yadava (2015) found theoretical expression for correlation coefficient between consecutive closed birth intervals in a heterogeneous population.

In their paper, Sheps et al.(1973) have made a distinction between closed birth interval and most recent closed birth interval. They have remarked that for a given age, the mean of the most recent closed birth interval in somewhat higher than the mean of usual closed birth interval based on cohort approach. For this, they have argued that this interval usually tends to select larger values more frequently resulting in the higher mean.

In this direction, Singh et al. (1988) developed a parity dependent model for most recent closed birth interval considering specific distributions for various components of the closed birth interval. Sharma (2004) studied the difference between usual closed birth interval and the most recent closed birth interval considering a specific probability distribution. He showed that if the fertility parameters are constant then there is no difference between usual closed birth interval and most recent closed birth interval. However, he showed that the mean of the most recent closed birth interval is somewhat higher than the mean of the usual closed birth interval if the population is heterogeneous with respect to non-susceptible period or conception rate or both. Later on, Kumar (2012) obtained expressions for the probability distribution of most recent closed birth interval and found expression for its mean and studied the mean of most recent closed birth interval and usual closed birth interval taking some specific hypothetical examples by specifying the values of marital duration T and the birth order.

Estimation of Parity Progression Ratios (PPR) and Instantaneous Parity Progression Ratios (IPPR) fromOpen and Closed birth interval data: The parity progression ratio (PPR) was introduced by Henry (1953) as a useful measure of fertility. It did not, however, gain wide application because of various difficulties encountered with its measurement, data needs, and conceptualization with respect to cohort and period measures. Srinivasan (1968) introduced the instantaneous parity progression ratio (IPPR) which is conceptually different from PPR. In fact, *PPR denotes the probability that a woman after delivering heri^{th} birth will ever proceed to the next birth, while IPPR is the probability that a women of parity $i$at the time of survey will ever proceed to the next birth.*

Srinivasan (1968) gave a procedure to estimate the IPPR for parity$i$ which requires data on T_i, the interval between the i^{th} and the $(i+1)^{\text{th}}$ birth and V_i, the interval between the i^{th} birth and terminal point of the reproductive period (say 45 years) for the females who have crossed the reproductive age and the i^{th} child happens to be their last child. While deriving his procedure, Srinivasan assumed that the i^{th} order births are uniformly distributed over time and showed that average open birth interval of fertile females (the females who definitely proceed to next birth) is $\frac{E[T_i^2]}{2\,E[T_i]}$ while the mean open birth interval for sterile females (the females who become sterile after giving i^{th} birth) is $\frac{E[V_i^2]}{2\,E[V_i]}$. However, usually data on V_i lack in most of the fertility surveys and if available, generally suffer from different types of biases.

In the light of above difficulty, Yadava and Bhattacharya (1985) proposed a procedure to estimate PPR utilizing data only on closed and open birth interval by considering only those females who have open birth interval (OBI) less than or equal to a specified period C, where C is so chosen that the probability of next birth from the birth of a child after C, is almost negligible. Later on Yadava and Saxena (1989) have given a methodology to convert PPR to IPPR and vice versa. The methodology converts PPR to IPPR by computing that if α_i is the proportion of females who proceed to next birth after giving their i^{th} birth then what would be this proportion at the time of the survey.

In Yadava and Bhattacharya (1985) procedure, the choice of C such that$P[Ti \geq C] \approx 0$ has been taken mainly to evaluate an integral $\int_0^\infty \left(1 - F_i(t)\right)dt$ as $\int_0^C \left(1 - F_i(t)\right)dt = E(T_i)$where $F_i(t)$ is the distribution function of T_i.Yadava et al. (1992) relaxed this condition and alternatively suggested that$\int_0^C \left(1 - F_i(t)\right)dt$can be evaluated for smaller values of Calso by using an appropriate quadrature formula provided the distribution of $T_i$is known.

One of the major assumptions in Srinivasan (1968); Yadava and Bhattacharya (1985); Yadava and Saxena (1989) and Yadava et al. (1992) is that the birth of i^{th} order are uniformly distributed over time. This assumption may be reasonably true for a stationary population with no change in fertility schedule over time. However for populations with changing pattern in fertility schedule, this assumption may not be appropriate. Recently Yadava et al. (2013b) have given a procedure for this situation and applied it to NFHS-3 data for states of Andhra Pradesh and Tamilnadu which experienced significant decline in fertility near the NFHS-3 survey.

Although we have mentioned above that PPR and IPPR are conceptually different but normally it is not easy to visualize the difference between the two. So we try to explain this difference mathematically as given below.

At the time of the survey, women of parity i may proceed to the next parity or may not. We refer these two types of women as “fertile” and “sterile”. The proportion of fertile and sterile women giving a i^{th} birth is α_i and $(1-\alpha_i)$ respectively. Obviously α_i is PPR for parity i. For a fertile woman, $T_i$denotes the interval between the i^{th} and the $(i+1)^{th}$ birth. The probability density of $T_i$is $f_i(t)$ and the cumulative distribution function is$F_i(t)$. If it is assumed as Srinivasan (1968) did, that i^{th} order births are uniformly distributed over time, then the total number of i^{th} order births during any time interval $(t, t+dt)$ is $B_i dt$(which means that it depends on the length dt of the interval but not on t) out of which $\alpha_i B_i dt$women will be fertile and $(1-\alpha_i)B_i dt$ will be sterile.The total number of fertile women of parity i having an open birth interval between t and $t+dt$ is $\alpha_i B_i(1-F_i(t))dt$, and then the total number of fertile women having an i^{th} order open birth interval at the time of the survey will be $\int_0^\infty \alpha_i B_i\big(1-F_i(t)\big)dt$if the duration of the reproductive period is assumed to be infinite. However, usually $\big(1-F_i(t)\big)$ becomes almost zero for a finite value of t, say C^* and consequently$\int_0^\infty \big(1-F_i(t)\big)dt = \int_0^{C^*}\big(1-F_i(t)\big)dt$.$T_i$being a positive random variable $\int_0^\infty \big(1-F_i(t)\big)dt = E(T_i)$,

The total number of fertile women having open birth interval less than C will be

$$\int_0^C \alpha_i B_i\big(1-F_i(t)\big)dt,$$

Because i^{th} order births are uniformly distributed over time and $(1-\alpha_i)$ is the proportion of females who become sterile after giving their i^{th} birth, the number of such sterile females will also be uniform over time. Thus the total number of sterile women with open birth interval less than C will be

$$\int_0^C (1-\alpha_i)B_i dt = (1-\alpha_i)B_i C\,.$$

So the total number of females at the time of survey with open birth interval less than C will be

$$\int_0^C \alpha_i B_i\big(1-F_i(t)\big)dt + (1-\alpha_i)B_i C$$

Thus the proportion of fertile women in the sample at the time of the survey with open birth interval less than C will be

$$\alpha_i{}^* = \frac{\int_0^C \alpha_i B_i\big(1-F_i(t)\big)dt}{\int_0^C \alpha_i B_i\big(1-F_i(t)\big)dt + (1-\alpha_i)B_i C} = \frac{\int_0^C \alpha_i\big(1-F_i(t)\big)dt}{\int_0^C \alpha_i\big(1-F_i(t)\big)dt + (1-\alpha_i)C}\,.$$

$$\alpha_i{}^* = \frac{\alpha_i E(T_i)}{\alpha_i E(T_i) + (1-\alpha_i)C}$$

This proportion $\alpha_i{}^*$ is known as IPPR for those women whose open birth interval is less than or equal to Cat the time of survey, which obviously differs fromα_ii.e. PPR. It does depend not only on α_i but also on $E(T_i)$ and C. Based on this approach, Yadava and Srivastava (1998) gave a methodology to estimate the proportion of fertile females among the females who have their open birth interval between $C_1$and C_2 (say).

It is pertinent to mention here that we have only given a description of the methods for computation of PPR and IPPR from data on open and closed birth interval. However, there exist other methods also for computation of PPR but we have not included those here. Although estimates of PPR for different parities are quite important and informative but at the same time these can also be used for computation of total marital fertility rate (TMFR) and total fertility rate (TFR). The TMFR is given as

$$\text{TMFR}=\alpha_0 + \alpha_0\alpha_1 + \alpha_0\alpha_1\alpha_2+\ldots$$

whereα_0 is the probability that a female after marriage will ever proceed to first birth. Similarly α_1 is the probability that a female at parity one will ever proceed to the second birth and so on. If we multiply the TMFR by the probability that a female will ever marry, we get the value of TFR.

Son preference and sex ratio at birth: Son preference is a worldwide phenomenon and perhaps it is more pronounced in Indian rural society than elsewhere. Many demographers believe that the strong desire for sons is one of the major causes for high fertility of rural Indian females. A natural question on this issue also arises: Does the son preference alter the sex ratio at birth (SRB)? Winston (1932) investigated this issue with the help of certain data and stated that the practice of son preference along with birth control may increase the SRB in favour of males. Later on, it was found that this conjecture was wrong, and that the behaviour of son preference will not alter the SRB in a population [Robbins (1952)]. Weiler (1959) also showed that son preference along with birth control will not alter the SRB. However, he has mentioned that this is true only when the probability of producing a male child (p) is the same for all females of a population, and that the SRB will get affected if p varies among females.

Goodman (1961) also showed that, generally, any stopping rule would have no effect on the SRB for constant p, but, if there is variation in p, the SRB, of course, does get affected. Sheps (1963) has derived an expression for the distribution of the size of families when the couples achieved the desired minimum family size and sex composition of the children. She considered the case of a predetermined maximum number of children to be born and used the strategy of at least b boys and g girls subject to the condition that $b + g \leq k$. Mitra (1970), on the other hand, used the strategy of having a minimum of b boys and g girls in not more than k trials. Both Sheps and Mitra assumed the probability of producing a male child to be constant for all couples. Keyfitz (1968) showed mathematically that the SRB in a population becomes smaller when couples use a male-preferring stopping rule if there exists variability in p. This holds true when we weight p according to a probability distribution $f(p)$ such that $\int_0^1 f(p)dp = 1$. For example, if couples stop child bearing once they get their first son, then the SRB will be the harmonic mean of p, which is always less than the average p.

The results derived by Weiler (1959), Goodman (1961) and Keyfitz (1968) for the case of heterogeneous females are purely theoretical in nature based on mathematical foundation. However, nothing definite is known about whether the probability of producing a male child is constant or whether it varies in a population of females. On this issue, Malinvaud (1955) and Garenne (2009), based on a large amount of data on the sex of the children born, have claimed that couples vary in their probability of producing a boy. Recently James (2011) has also agreed that there may be some biological reasons for heterogeneity in the probability of producing a male child. While examining the possible effect on the sex ratio of human births of the cycle day of conception, James (2000) has concluded that “there seems strong evidence that humans offspring sex ratio is reportedly associated with the cycle day of conception”. This may produce a variation in the probability of producing a male child in a population due to the variation in coital pattern during a menstrual cycle. However, all these results show a possibility of heterogeneity with respect to probability of producing a male child, but nothing definite is known on the possible variation in this probability. So an investigation of the effect of this variability on the SRB remains valid.

In all the above mentioned papers, the authors have assumed that females (or couples) do not have any control over the sex of the child and that the sex of the child is purely random in nature (of course they may have a preference for the sex of the child). However, this condition may change if couples have control over the sex of the child. This may happen if couples opt for sex selective abortions. This option along with sex preferred stopping rules may have more impact on the SRB. Recently Yadava et al. (2013a) have studied the impact of various hypothetical sex selective abortion programs on sex ratio at birth. Yadava et al. (2015) have tried to study the probability of coition on different days of menstrual cycle near the day of ovulation using the markov chain approach and tried to study its impact on sex ratio at birth.

It is pertinent to mention here that the present paper does not provide a comprehensive description of all the stochastic models developed for human fertility but only tries to give a brief description of the directions in which the stochastic models have been developed at the Department of Statistics, Banaras Hindu University.

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